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Advances on Asymptotic Stability of Impulsive Stochastic Evolution Equations

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Abstract

Asymptotic stability of impulsive stochastic evolution equations (ISEEs) with nonlocal initial conditions is studied. Stability analysis of impulsive stochastic differential equations with local initial conditions has received substantial attention. While relatively, little has been done on asymptotic stability with nonlocal initial conditions. Considering the advantage nonlocal initial value problems have over local initial value problems, we study the asymptotic stability of ISEEs with nonlocal initial conditions. Using the contraction mapping technique and by perturbing the local initial condition, asymptotic stability of ISEEs is established.

1 Introduction

Recently Impulsive evolution differential equations (stochastic, ordinary, partial, functional, quantum, etc.) with local and nonlocal initial conditions have become more relevant in so many mathematical models [1-23] of real processes and phenomena studied in virtually all branches of science, technology, social sciences, etc. There has been an appreciable development in impulse theory, particularly concerning impulsive differential equations with

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fixed moments [4-5, 9, 11, 14-15, 17-18, 20, 24-29, 32]. The study of the theory of stability of stochastic differential equations (SDEs) has been investigated by the Lyapunov functional method, energy estimates method, etc. (see [22, 30] and the references therein). The Lyapunov function has been used as a vital instrument to study the qualitative properties of the trivial solutions of differential equations [5, 11, 13, 15, 17]. Oleksiy [13] considered the asymptotic stability of a system of SDEs which has a zero solution by assuming the existence of a positive definite function. Wu et al. [4] introduced a time-varying discontinuous Lyapunov-Krasovskii functional to establish the stability of an impulsive system. The Lyapunov method, though extensively used, has some difficulties especially when studying asymptotic stability of SDEs (see [22] and the references therein). By applying some novel techniques, such as the fixed point approach, some of these difficulties have been addressed. Jing et al. [8] worked on the existence and exponential stability for a class of neutral stochastic partial differential equations using the Banach fixed point principle. Recently, Sufang et al. [9] investigated the existence and exponential stability of mild solutions for a class of impulsive stochastic differential equations (ISDEs) driven by a functional Brownian motion via Monch fixed point theorem. By using the fixed point approach Bishop et al. [10] discussed Ulam's type of stability of quantum stochastic differential equations (QSDEs) which led to some interesting results. Evolution differential equations have found practical applications in virtually all fields of science and technology [18, 21, 23, 31]. Also, problems with nonlocal initial conditions have been shown to have an advantage over traditional initial value problems ([9, 13, 29]). Motivated by the results in the References [6, 8, 9, 22, 23], we study the stability of solutions of ISEEs with nonlocal initial conditions. Hence, the results in this paper would generalize results on asymptotic stability of stochastic differential equations with initial local conditions and will have application to a wider class of equations. The rest of this paper consists of two sections: section 2 has to do with preliminary results, while the major results are discussed in section 3.

2 Preliminaries

Let (Ω, Σ, P) be a complete probability space, $\{\Sigma_t\}_{t\geq 0}$ a normal filtration. H, Y are separable Hilbert spaces with the usual norms $\|.\|_H$ and $\|.\|_Y$ respectively. Let Q be a Wiener process on (Ω, Σ, P) with tr$Q < \infty$. Let $L(H, Y)$ denote the space of linear operators that are bounded, $C(H, H)$ the space of

all continuous functions defined on H, $\Sigma_t = \Sigma_t^w$ a sigma algebra and L_2^0 is as defined in [22].
Of interest here is the following impulsive stochastic evolution equations (ISEEs):

$$\begin{aligned} d\phi(t) &= [A\phi(t) + F(t,\phi(t))]dt + G(t,\phi(t))dw(t) \\ & \quad t \in I = [0,T] \subseteq \mathbb{R}_+, t \neq t_k \\ & \quad \triangle\phi(t_k) = J_k(\phi(t_k)), t = t_k, k = 1, ..., m \\ & \quad \phi(0) = \phi_0 + g(\phi), t \in [0,T], 0 \leq T < \infty. \end{aligned} \tag{2.1}$$

Here $J_k \in C(H,H)$, A is defined as the infinitesimal generator of a semigroup of bounded linear operators $S(t), t \geq 0$, in H. The function $g : C([0,T],H) \to H$ is continuous and compact such that $E\{sup \|g(\phi)\|_H\} < +\infty$.
$\phi(t_k^+)$, $\phi(t_k^-)$ are the right and left limits of $\phi(t)$ at t_k, where $0 < t_1 < t_2 < ... < t_k < T$. $\Delta\phi(t_k) = \phi(t_k^+) - \phi(t_k^-)$ is the jump in the state ϕ at t_k. The maps $F \in (\mathbb{R}_+ \times H, H)$ and $G \in (\mathbb{R}_+ \times H, L(H,Y))$ are Borel measurable.
Definition 2.1. A stochastic process $\phi(t)$ is called a mild solution of (2.1) if:
1. $\phi(t)$ is Σ_t- adapted,
2. $\phi(t) \in H$ admits jumps for $t \in [0,T]$ and
3. $\phi(t)$ satisfies the integral equation

$$\begin{aligned} \phi(t) &= S(t)[\phi(0) + g(\phi)] \\ & \quad + \int_0^t S(t-s)\left(F(s,\phi(s))ds + G(s,\phi(s))dw(s)\right) \\ & \quad + \sum_{0<t_k<t} S(t-t_k)J_k(\phi(t_k)). \end{aligned} \tag{2.2}$$

Definition 2.2. Equation (2.2) is stable in the pth moment if for any arbitrary $\epsilon > 0$ we can find a $\delta > 0$ such that $\|\phi_0\|_H < \delta$ and
$E\left\{\sup_{t\geq 0} \|\phi(t)\|_H^p\right\} < \epsilon$, $p \geq 2 \in \mathbb{N}$.
Definition 2.3. Equation (2.2) is asymptotically stable in the pth moment if Definition 2.2 holds such that for any $\phi_0 \in H$,

$$\lim_{t\to\infty} E\left\{\sup_{t\in[0,T]} \|\phi(t)\|_H^p\right\} = 0.$$

We state the following well known results.
Lemma 2.4. For any $\gamma \geq 1$ and for arbitrary L_2^0- valued process $\vartheta(.)$,

$$\sup_{s\in[0,t]} E\left\|\int_0^s \vartheta(u)dw(u)\right\|_H^{2\gamma} \leq (\gamma(2\gamma-1))^\gamma\left(\int_0^t (E\|\vartheta(s)\|_{L_2^0}^{2\gamma})^{1/\gamma}ds\right)^\gamma.$$

Theorem 2.5. Let Ψ be a contraction operator on a Banach space B, then there exists a unique point $x \in B$ for which $\Psi(x) = x$.
Remark 2.6. The proof of these results can be found in [22] and the references therein.

3 Major Results

In this section, we make some assumptions on the nonlocal conditions and establish asymptotic stability of solutions of the given problem by transforming the problem into a fixed point problem. We show that the problem has a fixed point in the state space using the contraction mapping method and we are done.
We assume the following conditions:
1. $F(t,0) = 0$, $G(t,0) = 0$ and $J_k(0) = 0, k = 1,2,...,m$.
2. The zero solution of (2.1) exists when $\phi = 0$ and $g(0) = 0$.
3. The map $\psi(t,w) : [0,\infty) \times \Omega \to \mathbb{R}$ is continuous in t such that $E\|\psi(t,w)\|_H^p \to 0$ as $t \to \infty$.
Next, we state the following major conditions:
Let $c_k > 0$, $M \geq 0$, $L_g \geq 0$, K and $L > 0$ (are Lipschitz constants) such that
(S_1) for each $\phi, \varphi \in H$, $\|J_k(\phi) - J_k(\varphi)\| \leq c_k \|\phi - \varphi\|, k = 1,2,...,m$;
(S_2) $\|S(t)\|_H \leq Me^{-at}$, $t \geq 0, a > 0$;
(S_3) $\|g(\phi) - g(\varphi)\|_H \leq L_g \|\phi - \varphi\|_H$;
(S_4) $\|F(t,\phi) - F(t,\varphi)\|_H \leq K\|\phi - \varphi\|_H$, $\|G(t,\phi) - G(t,\varphi)\|_H \leq L\|\phi - \varphi\|_H$;
(S_5) $4^{p-1}M^p(L_g a^{-p} + K^p a^{-p} + L^p l_p (2a)^{-p/2} + L_k) < 1$,
where $l_p = (p(p-1)/2)^{p/2}$ and $L_k = e^{-apT}E\left(\sum_{k=1}^m \|c_k\|_H\right)$;
and
(S_6) for any arbitrary $\epsilon > 0$ and $\phi(t) \in H, t \in [0,T]$, $\|\phi(t)\|_H^p < \epsilon$.
Theorem 3.1. Assume that the conditions $(S_1) - (S_6)$ are satisfied. Then the mild solution of the nonlocal problem (2.1) is asymptotically stable in the pth moment.
Proof. Let the operator $\Gamma : C(I,B) \to C(I,B)$. We define a mapping Γ on

$C(I, B)$ by

$$\begin{aligned}\Gamma(\phi)(t) &= S(t)[\phi_0 + g(\phi)] + \int_0^t S(t-s)F(s,(s))ds \\ &+ \int_0^t S(t-s)G(s,\phi(s))ds + \sum_{0<t_k<t} S(t-t_k)J_k(\phi(t_k)) \\ &= \sum_{i=1}^{4} \Pi_i(t), t \geq 0, i = 1,2,3,4. \qquad (3.1)\end{aligned}$$

It suffices to show that Γ is a contraction operator on B (see [22] and the references therein). We show this in the following steps.
Step 1: The operator Γ is continuous.
Let $0 < \tau < \epsilon$. Then

$$E\left\|\Gamma(\phi)(t_1+\tau) - \Gamma(\phi)(t_1)\right\|_H^p \leq 4^{p-1} \sum_{i=1}^{4} E\left\|\Pi_i(t_1+\tau) - \Pi_i(t_1)\right\|_H^p.$$

and

$$E\left\|\Pi_i(t_1+\tau) - \Pi_i(t_1)\right\|_H^p \to 0$$

as $\tau \to 0, i = 1,2,3,4$.
Using the well known Holder's inequality and Lemma 2.4, we obtain

$$\begin{aligned}E\left\|\Pi_3(t_1+\tau) - \Pi_3(t_1)\right\|_H^p &\leq 3^{p-1} l_p \left[E\left\|((S(t_1+\tau) - S(t_1))g(\phi)\right\|_H^p)^{(2/p)}\right]^{(2/p)} \\ &+3^{p-1} l_p \left[\int_0^{t_1} (E\left\|(S(t_1+\tau-s) - S(t_1-s))G(s,\phi(s))\right\|_H^p)^{(2/p)} ds\right]^{(p/2)} \\ &+3^{p-1} l_p \left[\int_{t_1}^{t_1+\tau} (E\left\|S(t_1+\tau-s)G(s,\phi(s))\right\|_H^p)^{(2/p)} ds\right]^{(p/2)} \to 0 \qquad (3.2)\end{aligned}$$

as $\tau \to 0$.
Showing that the operator Γ is continuous on $[0, T]$.
Step 2: $\Gamma(B) \subset B$.
Using (3.1), gives

$$\begin{aligned}E\left\|(\Gamma\phi)(t)\right\|_H^p &\leq 4^{p-1} E\left\|S(t)[\phi_0 + g(\phi)]\right\|_H^p \\ &+4^{p-1} E\left\|\int_0^t S(t-s)F(s,\phi(s))ds\right\|_H^p \\ &+4^{p-1} E\left\|\int_0^t S(t-s)G(s,\phi(s))dws\right\|_H^p \\ &+4^{p-1} \sum_{0<t_k<t} E\left\|S(t-t_k)J_k(\phi(t_k))\right\|_H^p. \qquad (3.3)\end{aligned}$$

Using (S_1)-(S_3) yields

$$4^{p-1}E\left\|S(t)[\phi_0+g(\phi)]\right\|_H^p \leq M^p e^{-pat}\left\|\phi_0+g(\phi)\right\|_H^p \to 0 \tag{3.4}$$

and

$$4^{p-1}\sum_{0<t_k<t} E\left\|S(t-t_k)J_k(\phi(t_k))\right\|_H^p \leq M^p e^{-pat}\left\|J_k(\phi(t_k))\right\|_H^p \to 0 \tag{3.5}$$

as $t\to\infty$.
By using (S_1), (S_4) and Holder's inequality, we derive

$$\begin{aligned}4^{p-1}E\left\|\int_0^t S(t-s)F(s,\phi(s))ds\right\|_H^p &\leq 4^{p-1}M^pK^p\left[\int_0^t e^{-a(t-s)}ds\right]^{p-1}\\ &\quad\times\int_0^t e^{-a(t-s)}E\left\|\phi(s)\right\|_H^p ds\\ &\leq 4^{p-1}M^pK^pa^{1-p}\int_0^t e^{-a(t-s)}E\left\|\phi(s)\right\|_H^p ds.\end{aligned} \tag{3.6}$$

Now by (S_6) and (3.4) we get

$$\begin{aligned}4^{p-1}E\left\|\int_0^t S(t-s)F(s,\phi(s))ds\right\|_H^p &\leq 4^{p-1}M^pK^pa^{1-p}e^{-at}\int_0^{t_1} e^{as}E\left\|\phi(s)\right\|_H^p ds\\ &\quad+4^{p-1}M^pK^pa^{-p}\epsilon.\end{aligned} \tag{3.7}$$

By the hypothesis of Theorem 3.1 as $t\to\infty$, $e^{-at}\to 0$. Now, for $0<t_1<t$, we obtain

$$4^{p-1}M^pK^pa^{1-p}e^{-at}\int_0^{t_1} e^{as}E\left\|\phi(s)\right\|_H^p ds \leq \epsilon - 4^{p-1}M^pK^pa^{-p}\epsilon. \tag{3.8}$$

Hence, using (3.7) and (3.8), yields

$$4^{p-1}E\left\|\int_0^t S(t-s)F(s,\phi(s))ds\right\|_H^p < \epsilon.$$

Which implies that

$$4^{p-1}E\left\|\int_0^t S(t-s)F(s,\phi(s))ds\right\|_H^p \to 0 \text{ as} t\to\infty. \tag{3.9}$$

Similarly, for $\phi \in B$ and $t \in [0,T]$,

$$4^{p-1}E\left\|\int_0^t S(t-s)G(s,\phi(s))dw(s)\right\|_H^p \leq 4^{p-1}l_pM^pL^p \times\left[\int_0^t e^{-2a(t-s)}(E\left\|\phi(s)\right\|_H^p)^{2/p}ds\right]^{p/2}. \tag{3.10}$$

So that in the same vain as (3.9) and (3.10), we get

$$4^{p-1}E\left\|\int_0^t S(t-s)G(s,\phi(s))dw(s)\right\|_H^p \to 0 \text{ as t} \to \infty. \tag{3.11}$$

Therefore, substituting (3.4), (3.5), (3.9) and (3.11) in (3.3), we see that $E\left\|(\Gamma\phi)(t)\right\|_H^p \to 0$ as $t \to \infty$. Hence, $\Gamma(B) \subset B$.
Step 3: The operator Γ is a contraction mapping on B.
For $\phi, \varphi \in B$, $t \in [0,T]$,

$$\begin{aligned}\sup_{s\in[0,T]} E\left\|(\Gamma\phi)(t)-(\Gamma\varphi)(t)\right\|_H^p \leq\ & 4^{p-1}E\left\|S(t)[g(\phi)-g(\varphi)]\right\|_H^p \\ & +4^{p-1}\sup_{s\in[0,T]} E\left\|\int_0^t S(t-s)[F(s,\phi(t))-F(s,\varphi(s))ds]\right\|_H^p \\ & +4^{p-1}\sup_{s\in[0,T]} E\left\|\int_0^t S(t-s)[G(s,\phi(t))-G(s,\varphi(s))dw(s)]\right\|_H^p \\ & +4^{p-1}\sup_{s\in[0,T]} E\left\|\sum_{0<t_k<t} S(t-t_k)[J_k(\phi(t_k))-J_k(\varphi(t_k))]\right\|_H^p \\ & \leq [4^{p-1}M^p(L_ga^{-p}+K^pa^{-p}+L^pl_p(2a)^{-p/2}+L_k)] \\ & \times\left(\sup_{t\in[0,T]} E\left\|\phi(t)-\varphi(t)\right\|_H^p\right). \end{aligned} \tag{3.12}$$

Therefore,

$$\sup_{s\in[0,T]} E\left\|(\Gamma\phi)(t)-(\Gamma\varphi)(t)\right\|_H^p \leq \sup_{t\in[0,T]} E\left\|\phi(t)-\varphi(t)\right\|_H^p,$$

where $4^{p-1}M^p(L_ga^{-p}+K^pa^{-p}+L^pl_p(2a)^{-p/2}+L_k) < 1$ showing that Γ is a contraction mapping. This implies that a fixed point $\phi(.) \in B$ which is a solution of the nonlocal problem (2.1) exists and $E\left\|\phi(t)\right\|_H^p \to 0$ as $t \to \infty$.

The proof is therefore complete and by Definition 2.3 we conclude that the solution of the nonlocal problem (2.1) is asymptotically stable.

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