Modified Weibull Distribution: Ordinary Differential Equations

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Abstract— **Modified Weibull distribution is an appreciable improvement over the Weibull distribution. This paper explores the application of differentiation to obtain the ordinary differential equations (ODE) of the probability functions of the modified Weibull Distribution. The parameters and support that characterized the distribution inevitably determine the behavior, existence, uniqueness and solution of the ODEs. The method is recommended to be applied to other probability distributions and probability functions not considered in this paper. Computer codes and programs can be used for the implementation.**

Index Terms— **Differential calculus, quantile function, hazard function, reversed hazard function, survival function, inverse survival function, probability density function, Weibull.**

I. Introduction

CALCULUS in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-4].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of distributions can be transformed as ODE whose solution yields the respective PDF. Some of which are available: see [5-9].

The aim of this paper is to propose homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the modified Weibull distribution. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the conditions necessary for the existence of the ODEs. Similar results for other distributions have been proposed, see [10-23] for details.

The modified Weibull distribution considered was the one proposed by [24-26] as a generalization of the parent Weibull, exponential, Rayleigh and linear hazard rate distributions. Since the introduction of the distribution as a lifetime model, several researchers have studied different aspects of the distribution. Gasmi and Berzig [27] worked on parameter estimation based on type I censored samples, Soliman et al. [28] applied Markov Chain Monte-Carlo approach on progressive censored data while Jiang et al. [29] done their estimation based on type II censored samples. There are other modified Weibull models such as the ones proposed by [30] and [31].

The distribution has been used in the development of new models which includes; modified inverse Weibull distribution [32], transmuted modified inverse Weibull distribution [33], beta transmuted Weibull distribution [34], the modified Weibull geometric distribution [35] and Weibull exponential distribution [36-37].
Differential calculus was used to obtain the results.

II. Probability Density Function

The PDF of the modified Weibull distribution is given as;

$$f(x) = (\alpha + \beta\gamma x^{\gamma-1})\,e^{-\alpha x - \beta x^{\gamma}} \quad (1)$$

Differentiate equation (1);

$$f'(x) = \left\{ \frac{\beta\gamma(\gamma-1)x^{\gamma-2}}{\alpha + \beta\gamma x^{\gamma-1}} - (\alpha + \beta\gamma x^{\gamma-1}) \right\} f(x) \quad (2)$$

The equation can only exists for $\alpha, \beta, \gamma, x > 0$.

The ODEs can be obtained for any given values of α, β and γ.

When $\gamma = 1$, equation (2) becomes;

$$f_d'(x) = -(\alpha + \beta) f_d(x) \quad (3)$$

$$f_d'(x) + (\alpha + \beta) f_d(x) = 0 \quad (4)$$

When $\gamma = 2$, equation (2) becomes;

$$f_e'(x) = \left\{ \frac{2\beta}{\alpha + 2\beta x} - (\alpha + 2\beta x) \right\} f_e(x) \quad (5)$$

$$(\alpha + 2\beta x) f_e'(x) - (2\beta - (\alpha + 2\beta x)^2) f_e(x) = 0 \quad (6)$$

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When $\gamma = 3$, equation (2) becomes;

$$f_f'(x) = \left\{ \frac{6\beta x}{\alpha + 3\beta x^2} - (\alpha + 3\beta x^2) \right\} f_f(x) \tag{7}$$

$$(\alpha + 3\beta x^2) f_f'(x) - (6\beta x - (\alpha + 3\beta x^2)^2) f_f(x) = 0 \tag{8}$$

Equation (2) is differentiated;

$$f''(x) = \left\{ \begin{array}{l} -\dfrac{(\beta\gamma(\gamma-1)x^{\gamma-2})^2}{(\alpha + \beta\gamma x^{\gamma-1})^2} \\ +\dfrac{\beta\gamma(\gamma-1)(\gamma-2)x^{\gamma-3}}{(\alpha + \beta\gamma x^{\gamma-1})} \\ -\beta\gamma(\gamma-1)x^{\gamma-2}) \end{array} \right\} f(x) + \left\{ \frac{\beta\gamma(\gamma-1)x^{\gamma-2}}{\alpha + \beta\gamma x^{\gamma-1}} - (\alpha + \beta\gamma x^{\gamma-1}) \right\} f'(x) \tag{9}$$

The equation can only exists for $\alpha, \beta, \gamma, x > 0$.

These presented equations from (2) are needed to further simplification of equation (9);

$$\frac{\beta\gamma(\gamma-1)x^{\gamma-2}}{\alpha + \beta\gamma x^{\gamma-1}} - (\alpha + \beta\gamma x^{\gamma-1}) = \frac{f'(x)}{f(x)} \tag{10}$$

$$\frac{\beta\gamma(\gamma-1)x^{\gamma-2}}{\alpha + \beta\gamma x^{\gamma-1}} = \frac{f'(x)}{f(x)} + (\alpha + \beta\gamma x^{\gamma-1}) \tag{11}$$

$$\frac{\beta\gamma(\gamma-1)(\gamma-2)x^{\gamma-3}}{\alpha + \beta\gamma x^{\gamma-1}} = \frac{\gamma-2}{x}\left(\frac{f'(x)}{f(x)} + (\alpha + \beta\gamma x^{\gamma-1}) \right) \tag{12}$$

$$\left(\frac{\beta\gamma(\gamma-1)x^{\gamma-2}}{\alpha + \beta\gamma x^{\gamma-1}} \right)^2 = \left(\frac{f'(x)}{f(x)} + (\alpha + \beta\gamma x^{\gamma-1}) \right)^2 \tag{13}$$

Substitute equations (10), (12) and (13) into equation (9);

$$f''(x) = \frac{f'^2(x)}{f(x)} - f(x)\left(\frac{f'(x)}{f(x)} + (\alpha + \beta\gamma x^{\gamma-1}) \right)^2 + \frac{\gamma-2}{x}\left(\frac{f'(x)}{f(x)} + (\alpha + \beta\gamma x^{\gamma-1}) \right) f(x) - \beta\gamma(\gamma-1)x^{\gamma-2}) f(x) \tag{14}$$

$$f(1) = (\alpha + \beta\gamma) e^{-(\alpha+\beta)} \tag{15}$$

$$f'(1) = (\beta\gamma(\gamma-1) - (\alpha + \beta\gamma)^2) e^{-(\alpha+\beta)} \tag{16}$$

When $\gamma = 1$, equation (14) becomes;

$$f''(x) = -\left(\frac{f'^2(x)}{f(x)} + 2(\alpha+\beta) f'(x) + (\alpha+\beta)^2 f(x) \right) - \left(\frac{f'(x)}{x} + \frac{(\alpha+\beta) f(x)}{x} \right) + \frac{f'^2(x)}{f(x)} \tag{17}$$

$$f''(x) + (2(\alpha+\beta)x + 1) f'(x) + ((\alpha+\beta)^2 x + (\alpha+\beta)) f(x) = 0 \tag{18}$$

See [10-23] for details.

III. Quantile Function

The QF of the modified Weibull distribution is given as;

$$\alpha Q(p) + \beta Q^{\gamma}(p) = -\ln(1-p) \tag{19}$$

Equation (19) is differentiated in order to obtain the first order ODE;

$$\alpha Q'(p) + \beta\gamma Q^{\gamma-1}(p) Q'(p) = \frac{1}{1-p} \tag{20}$$

The equation can only exists for $0 \le p < 1$.

$$(1-p)(\alpha + \beta\gamma Q^{\gamma-1}(p)) Q'(p) - 1 = 0 \tag{21}$$

The ODE and their initial conditions can be derived for any given values of α, β and γ.

When $\gamma = 1$, equation (19) and (22) become;

$$(1-p)(\alpha+\beta) Q'(p) - 1 = 0 \tag{22}$$

$$\alpha Q(0) + \beta Q(0) = 0 \tag{23}$$

$$Q(0) = -(\alpha+\beta) \tag{24}$$

When $\gamma = 2$, equation (19) and (22) become;

$$(1-p)(\alpha + 2\beta Q(p)) Q'(p) - 1 = 0 \tag{25}$$

$$\alpha Q(0) + \beta Q^2(0) = 0 \tag{26}$$

$$Q(0) = -\frac{\alpha}{\beta} \tag{27}$$

When $\gamma = 3$, equation (19) and (22) becomes;

$$(1-p)(\alpha + 3\beta Q^2(p)) Q'(p) - 1 = 0 \tag{28}$$

$$\alpha Q(0) + \beta Q^3(0) = 0 \tag{29}$$

$$Q(0) = \sqrt{\frac{\alpha}{\beta}} i \tag{30}$$

Differentiate equation (20), to obtain the second order ODE;

$$\alpha Q''(p) + \beta\gamma Q^{\gamma-1}(p) Q''(p) + \beta\gamma(\gamma-1) Q^{\gamma-2}(p) Q'(p) = -\frac{1}{(1-p)^2} \tag{31}$$

The equation can only exists for $0 \le p < 1$.

The ODEs and their initial conditions can only be derived for any given values of α, β and γ.

When $\gamma = 1$, equation (31) and (20) become;

$$\alpha Q''(p) + \beta Q''(p) + \beta\gamma = -\frac{1}{(1-p)^2} \tag{32}$$

$$\alpha Q'(0) + \beta Q'(0) = 1 \tag{33}$$

$$Q'(0) = \frac{1}{\alpha+\beta} \tag{34}$$

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When $\gamma = 2$, equation (31) and (20) becomes;

$$(\alpha + 2\beta Q(p))Q''(p) + 2\beta Q'(p) = -\frac{1}{(1-p)^2} \quad (35)$$

$$\alpha Q'(0) + 2\beta Q(0)Q'(0) = 1 \quad (36)$$

Using equation (27) in (36)

$$\alpha Q'(0) + 2\beta\left(-\frac{\alpha}{\beta}\right)Q'(0) = 1 \quad (37)$$

$$Q'(0) = -\frac{1}{\alpha} \quad (38)$$

When $\gamma = 3$, equation (31) and (20) becomes;

$$(\alpha + 3\beta Q^2(p))Q''(p) + 6\beta Q(p)Q'(p) = -\frac{1}{(1-p)^2} \quad (39)$$

$$\alpha Q'(0) + 3\beta Q^2(0)Q'(0) = 1 \quad (40)$$

Using equation (30) in (40)

$$\alpha Q'(0) + 3\beta\left(-\frac{\alpha}{\beta}\right)Q'(0) = 1 \quad (41)$$

$$Q'(0) = -\frac{1}{2\alpha} \quad (42)$$

See [10-23] for details.

IV. Survival Function

The SF of the modified Weibull distribution is given as;

$$S(t) = \mathrm{e}^{-\alpha t - \beta t^{\gamma}} \quad (43)$$

Differentiate equation (43);

$$S'(t) = -(\alpha + \beta\gamma t^{\gamma-1})\mathrm{e}^{-\alpha t - \beta t^{\gamma}} \quad (44)$$

The equation can only exists for $\alpha, \beta, \gamma, t > 0$.

$$S'(t) = -(\alpha + \beta\gamma t^{\gamma-1})S(t) \quad (45)$$

The ODEs can only be derived for any given values of α, β and γ.

When $\gamma = 1$, equation (45) becomes;

$$S_g'(t) = -(\alpha + \beta)S_g(t) \quad (46)$$

$$S_g'(t) + (\alpha + \beta)S_g(t) = 0 \quad (47)$$

When $\gamma = 2$, equation (45) becomes;

$$S_h'(t) = -(\alpha + 2\beta t)S_h(t) \quad (48)$$

$$S_h'(t) + (\alpha + 2\beta t)S_h(t) = 0 \quad (49)$$

When $\gamma = 3$, equation (45) becomes;

$$S_i'(t) = -(\alpha + 3\beta t^2)S_i(t) \quad (50)$$

$$S_i'(t) + (\alpha + 3\beta t^2)S_i(t) = 0 \quad (51)$$

Differentiate equation (45);

$$S''(t) = -\left((\alpha + \beta\gamma t^{\gamma-1})S'(t) + \beta\gamma(\gamma-1)t^{\gamma-2}S(t)\right) \quad (52)$$

These equations derived from (45) are required in the simplification of equation (52);

$$-(\alpha + \beta\gamma t^{\gamma-1}) = \frac{S'(t)}{S(t)} \quad (53)$$

$$-\beta\gamma t^{\gamma-1} = \frac{S'(t)}{S(t)} + \alpha \quad (54)$$

$$-\beta\gamma(\gamma-1)t^{\gamma-2} = \frac{\gamma-1}{t}\left[\frac{S'(t)}{S(t)} + \alpha\right] \quad (55)$$

Substitute equations (156) and (55) into equation (52);

$$S''(t) = \frac{S'^2(t)}{S(t)} + \frac{\gamma-1}{t}\left[\frac{S'(t)}{S(t)} + \alpha\right]S(t) \quad (56)$$

$$S''(t) = \frac{S'^2(t)}{S(t)} + \frac{(\gamma-1)S'(t)}{t} + \frac{(\gamma-1)\alpha S(t)}{t} \quad (57)$$

The second order ODE for the SF of the modified Weibull distribution is given by;

$$tS(t)S''(t) - tS'^2(t) - (\gamma-1)S(t)S'(t) - (\gamma-1)\alpha S^2(t) = 0 \quad (58)$$

$$S(1) = \mathrm{e}^{-(\alpha+\beta)} \quad (59)$$

$$S'(1) = -(\alpha + \beta\gamma)\mathrm{e}^{-(\alpha+\beta)} \quad (60)$$

See [10-23] for details.

V. Inverse Survival Function

The ISF of the modified Weibull distribution is given as;

$$p = \mathrm{e}^{-\alpha Q(p) - \beta Q^{\gamma}(p)} \quad (61)$$

$$-(\alpha Q(p) + \beta Q^{\gamma}(p)) = \ln p \quad (62)$$

Differentiate equation (62), to obtain the first order ODE;

$$\alpha Q'(p) + \beta\gamma Q^{\gamma-1}(p)Q'(p) = -\frac{1}{p} \quad (63)$$

The equation can only exists for $0 < p \le 1$.

$$(\alpha + \beta\gamma Q^{\gamma-1}(p))Q'(p) = -\frac{1}{p} \quad (64)$$

$$p(\alpha + \beta\gamma Q^{\gamma-1}(p))Q'(p) + 1 = 0 \quad (65)$$

The ODEs and their initial conditions can be derived for any given values of α, β and γ. Some examples are shown in Table 1.

Table 1: ODE of ISF for some selected parameters of the distribution

γ	β	α	Ordinary differential equation
1	1	1	$2pQ'(p) + 1 = 0$
1	1	2	$3pQ'(p) + 1 = 0$
1	2	1	$3pQ'(p) + 1 = 0$
1	2	2	$4pQ'(p) + 1 = 0$
2	1	1	$p(2Q(p) + 1)Q'(p) + 1 = 0$
2	1	2	$2p(Q(p) + 1)Q'(p) + 1 = 0$

See [10-23] for details.

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VI. Hazard Function

The HF of the modified Weibull distribution is given as;

$$h(t)=\alpha+\beta\gamma t^{\gamma-1} \tag{66}$$

Differentiate equation (66);

$$h'(t)=\beta\gamma(\gamma-1)t^{\gamma-2} \tag{67}$$

The equation can only exists for $\alpha,\beta,\gamma,t>0$.

Using equation (66) to simplify equation (67), equation (66) can also be written as;

$$\beta\gamma t^{\gamma-1}=h(t)-\alpha \tag{68}$$

$$\beta\gamma(\gamma-1)t^{\gamma-2}=\frac{\gamma-1}{t}(h(t)-\alpha) \tag{69}$$

Substituteequation (69) into equation (67);

$$h'(t)=\frac{\gamma-1}{t}(h(t)-\alpha) \tag{70}$$

The first order ODE for the HF of the modified Weibull distribution is given by;

$$th'(t)-(\gamma-1)(h(t)-\alpha)=0 \tag{71}$$

$$h(1)=\alpha+\beta\gamma \tag{72}$$

Differentiate equation (67);

$$h''(t)=\beta\gamma(\gamma-1)(\gamma-2)t^{\gamma-3} \tag{73}$$

The equation can only exists for $\alpha,\beta,\gamma,t>0$.

Two ODEs can be derived from equation (73);

ODE 1; Use equation (66) in equation (73);

$$t^2h''(t)=\beta\gamma(\gamma-1)(\gamma-2)t^{\gamma-1} \tag{74}$$

$$t^2h''(t)=(\gamma-1)(\gamma-2)(h(t)-\alpha) \tag{75}$$

$$t^2h''(t)-(\gamma-1)(\gamma-2)(h(t)-\alpha)=0 \tag{76}$$

ODE 2; Use equation (67) in equation (73);

$$th''(t)=\beta\gamma(\gamma-1)(\gamma-2)t^{\gamma-2} \tag{77}$$

$$th''(t)=(\gamma-2)h'(t) \tag{78}$$

$$th''(t)-(\gamma-2)h'(t)=0 \tag{79}$$

Differentiate equation (73);

$$h'''(t)=\beta\gamma(\gamma-1)(\gamma-2)(\gamma-3)t^{\gamma-4} \tag{80}$$

The equation can only exists for $\alpha,\beta,\gamma,t>0$.

Three ODEs can be derived from the further evaluation of equation (80);

ODE 1; Use equation (66) in equation (80);

$$t^3h'''(t)=\beta\gamma(\gamma-1)(\gamma-2)(\gamma-3)t^{\gamma-1} \tag{81}$$

$$t^3h'''(t)=(\gamma-1)(\gamma-2)(\gamma-3)(h(t)-\alpha) \tag{82}$$

$$t^3h'''(t)-(\gamma-1)(\gamma-2)(\gamma-3)(h(t)-\alpha)=0 \tag{83}$$

ODE 2; Use equation (67) in equation (80);

$$t^2h'''(t)=\beta\gamma(\gamma-1)(\gamma-2)(\gamma-3)t^{\gamma-2} \tag{84}$$

$$t^2h'''(t)=(\gamma-2)(\gamma-3)h'(t) \tag{85}$$

$$t^2h'''(t)-(\gamma-2)(\gamma-3)h'(t)=0 \tag{86}$$

ODE 3; Use equation (73) in equation (80);

$$th'''(t)=\beta\gamma(\gamma-1)(\gamma-2)(\gamma-3)t^{\gamma-3} \tag{87}$$

$$th'''(t)=(\gamma-3)h''(t) \tag{88}$$

$$th'''(t)-(\gamma-3)h''(t)=0 \tag{89}$$

$$h'(0)=0 \tag{90}$$

$$h''(0)=0 \tag{91}$$

See [10-23] for details.

VII. Reversed Hazard Function

The RHF of the modified Weibull distribution is given as;

$$j(t)=\frac{(\alpha+\beta\gamma t^{\gamma-1})\,\mathrm{e}^{-\alpha t-\beta t^{\gamma}}}{1-\mathrm{e}^{-\alpha t-\beta t^{\gamma}}} \tag{92}$$

Differentiate equation (92), to obtain the first order ODE;

$$j'(t)=\left\{\begin{matrix}\dfrac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}-(\alpha+\beta\gamma t^{\gamma-1})\\ -\dfrac{(\alpha+\beta\gamma t^{\gamma-1})\,\mathrm{e}^{-\alpha t-\beta t^{\gamma}}(1-\mathrm{e}^{-\alpha t-\beta t^{\gamma}})^{-2}}{(1-\mathrm{e}^{-\alpha t-\beta t^{\gamma}})^{-1}}\end{matrix}\right\}j(t) \tag{93}$$

The equation can only exists for $\alpha,\beta,\gamma,t>0$.

$$j'(t)=\left\{\begin{matrix}\dfrac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}-(\alpha+\beta\gamma t^{\gamma-1})\\ -\dfrac{(\alpha+\beta\gamma t^{\gamma-1})\,\mathrm{e}^{-\alpha t-\beta t^{\gamma}}}{(1-\mathrm{e}^{-\alpha t-\beta t^{\gamma}})}\end{matrix}\right\}j(t) \tag{94}$$

$$j'(t)=\left\{\frac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}-(\alpha+\beta\gamma t^{\gamma-1})-j(t)\right\}j(t) \tag{95}$$

Equation (95) is further differentiated;

$$j''(t)=\left\{\begin{matrix}-\dfrac{(\beta\gamma(\gamma-1)t^{\gamma-2})^2}{(\alpha+\beta\gamma t^{\gamma-1})^2}\\ +\dfrac{\beta\gamma(\gamma-1)(\gamma-2)t^{\gamma-3}}{(\alpha+\beta\gamma t^{\gamma-1})}\\ -\beta\gamma(\gamma-1)t^{\gamma-2})-j'(t)\end{matrix}\right\}j(t)$$
$$+\left\{\frac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}-(\alpha+\beta\gamma t^{\gamma-1})-j(t)\right\}j'(t) \tag{96}$$

The equation can only exists for $\alpha,\beta,\gamma,t>0$.

These equations derived from (95) are required to further evaluate equation (96);

$$\frac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}-(\alpha+\beta\gamma t^{\gamma-1})-j(t)=\frac{j'(t)}{j(t)} \tag{97}$$

$$\frac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}=\frac{j'(t)}{j(t)}+(\alpha+\beta\gamma t^{\gamma-1})+j(t) \tag{98}$$

$$\frac{\beta\gamma(\gamma-1)(\gamma-2)t^{\gamma-3}}{\alpha+\beta\gamma t^{\gamma-1}} = \frac{\gamma-2}{t}\left(\frac{j'(t)}{j(t)}+(\alpha+\beta\gamma t^{\gamma-1})+j(t)\right) \quad (99)$$

$$\left(\frac{\beta\gamma(\gamma-1)t^{\gamma-2}}{\alpha+\beta\gamma t^{\gamma-1}}\right)^2 = \left(\frac{j'(t)}{j(t)}+(\alpha+\beta\gamma t^{\gamma-1})-j(t)\right)^2 \quad (100)$$

Substitute equations (97), (99) and (100) into equation (96);

$$j''(t) = \frac{j'^2(t)}{j(t)} - j(t)\left(\frac{j'(t)}{j(t)}+(\alpha+\beta\gamma t^{\gamma-1})-j(t)\right)^2 + \frac{\gamma-2}{t}\left(\frac{j'(t)}{j(t)}+(\alpha+\beta\gamma t^{\gamma-1})+j(t)\right)j(t) \quad (101)$$

See [10-23] for details.

VIII. CONCLUDING REMARKS

Ordinary differential equations (ODEs) has been obtained for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of modified Weibull distribution. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. Different classes of the differential equations can be obtained for the different values of the parameters that defined the distribution. The parameter and the supports that characterize the distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [38-50]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

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