Classes of Ordinary Differential Equations Obtained for the Probability Functions of Gumbel Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODEs) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the Gumbel distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions, functions and can serve an alternative to approximation and estimation.**

***Index Terms*— Survival function, Gumbel distribution, hazard function, calculus, differentiation, probability density function**

I. Introduction

Gumbel distribution is often used in modeling the distribution of the minimum and maximum of different distributions. The distribution was proposed by Gumbel [1-2] and had undergone modifications such as its generalization [3-4], beta Gumbel distribution [5], exponentiated Gumbel distribution [6], Kumaraswamy Gumbel distribution [7], exponentiated generalized Gumbel distribution [8], McDonald Gumbel distribution [9] and transmuted exponentiated Gumbel distribution [10]. Some aspects of the distribution studied by several authors which include: Bayesian analysis [11] and interval estimation [12].

The distribution has been applied in different fields and areas such as: modeling annual distribution of flood [13-14], fitting extreme wind speeds [15-17], modeling and predicting storm [18], modeling the frequency of earthquakes [19], extreme rainfall data analysis by [20-22], estimate the probability of pipe wall perforation [23], extreme tsunami heights [24], irrigation analysis [25], estimation of the mean weight of fish in aquaculture cages [26], modeling and estimating risk of disease transmission [27] and modeling corrosion [28].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Gumbel distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [29], beta distribution [30], raised cosine distribution [31], Lomax distribution [32], beta prime distribution or inverted beta distribution [33].

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II. Probability Density Function

The probability density function of the Gumbel distribution is given as;

$$f(x)=\frac{1}{\sigma}\exp\left\{-\left[\frac{x-\mu}{\sigma}+\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\} \tag{1}$$

To obtain the first order ordinary differential equation for the probability density function of the Gumbel distribution, differentiate equation (1), to obtain;

$$f'(x)=-\frac{1}{\sigma^2}\left(1-\exp\left(-\frac{x-\mu}{\sigma}\right)\right) \exp\left\{-\left[\frac{x-\mu}{\sigma}+\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\} \tag{2}$$

The condition necessary for the existence of equation is $\sigma>0, \mu, x \in \mathbb{R}$.

Simplify using equation (1);

$$f'(x)=-\frac{1}{\sigma}\left(1-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)f(x) \tag{3}$$

Differentiating equation (3) leads to

$$f''(x) = -\frac{1}{\sigma}\left\{\left(1-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)f'(x) + \frac{1}{\sigma}\left(\exp\left(-\frac{x-\mu}{\sigma}\right)\right)f(x)\right\} \tag{4}$$

The condition necessary for the existence of equation is $\sigma > 0, \mu, x \in \mathbb{R}.$

Equation (3) can be written as;

$$\frac{f'(x)}{f(x)} = -\frac{1}{\sigma}\left(1-\exp\left(-\frac{x-\mu}{\sigma}\right)\right) \tag{5}$$

$$-\sigma\frac{f'(x)}{f(x)} = \left(1-\exp\left(-\frac{x-\mu}{\sigma}\right)\right) \tag{6}$$

$$\exp\left(-\frac{x-\mu}{\sigma}\right) = 1+\sigma\frac{f'(x)}{f(x)} \tag{7}$$

Substituting equations (6) and (7) into equation (4) gives

$$f''(x) = -\frac{1}{\sigma}\left\{-\sigma\frac{f'(x)}{f(x)}f'(x) + +\frac{1}{\sigma}\left(1+\sigma\frac{f'(x)}{f(x)}\right)f(x)\right\} \tag{8}$$

$$f''(x) = \left\{\frac{f'^2(x)}{f(x)} - \frac{f(x)}{\sigma^2}\left(1+\sigma\frac{f'(x)}{f(x)}\right)\right\} \tag{9}$$

$$f''(x) = \frac{f'^2(x)}{f(x)} - \frac{f'(x)}{\sigma} - \frac{f(x)}{\sigma^2} \tag{10}$$

The second order ordinary differential equation for the probability density function of the Gumbel distribution is given by;

$$\sigma^2 f(x)f''(x) - \sigma^2 f'^2(x) + \sigma f(x)f'(x) + f^2(x) = 0 \tag{11}$$

$$f(0) = \frac{1}{\sigma}\exp\left\{-\left[-\frac{\mu}{\sigma}+\exp\left(\frac{\mu}{\sigma}\right)\right]\right\} \quad (12$$

$$f'(0) = -\frac{1}{\sigma^2}\left(1-\exp\left(\frac{\mu}{\sigma}\right)\right)\left\{\exp\left\{-\left[-\frac{\mu}{\sigma}+\exp\left(\frac{\mu}{\sigma}\right)\right]\right\}\right\} \tag{13}$$

III. Quantile Function

The Quantile function of the Gumbel distribution is given as;

$$Q(p) = \mu - \sigma\ln(-\ln p) \tag{14}$$

To obtain the first order ordinary differential equation for the Quantile function of the Gumbel distribution, differentiate equation (14), to obtain;

$$Q'(p) = -\frac{\sigma}{p\ln p} \tag{15}$$

The condition necessary for the existence of equation is $\sigma > 0, 0 < p < 1.$

Differentiate equation (15), to obtain;

$$Q''(p) = \left[\frac{\sigma}{p^2(\ln p)^2} + \frac{\sigma}{p^2\ln p}\right] \tag{16}$$

The condition necessary for the existence of equation is $\sigma > 0, 0 < p < 1.$

Squaring both sides of equation (15), one obtains

$$Q'^2(p) = \frac{\sigma^2}{p^2(\ln p)^2} \tag{17}$$

$$\frac{Q'^2(p)}{\sigma} = \frac{\sigma}{p^2(\ln p)^2} \tag{18}$$

Also dividing both sides of equation (15) by p ;

$$\frac{Q'(p)}{p} = -\frac{\sigma}{p^2\ln p} \tag{19}$$

Substituting equations (18) and (19) into equation (16) gives

$$Q''(p) = \left[\frac{Q'^2(p)}{\sigma} - \frac{Q'(p)}{p}\right] \tag{20}$$

The second order ordinary differential equation for the Quantile function of the Gumbel distribution is given by;

$$\sigma pQ''(p) - pQ'^2(p) + \sigma Q'(p) = 0 \tag{21}$$

$$Q(0.1) = \mu - 0.834\sigma \tag{22}$$

$$Q'(0.1) = 4.343\sigma \tag{23}$$

IV. Survival Function

The survival function of the Gumbel distribution is given as;

$$S(t) = 1-\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\} \tag{24}$$

To obtain the first order ordinary differential equation for the survival function of the Gumbel distribution, differentiate equation (24), to obtain;

$$S'(t) = -\frac{1}{\sigma}\left(\exp\left(-\frac{t-\mu}{\sigma}\right)\right)\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\} \tag{25}$$

The condition necessary for the existence of equation is $\sigma > 0, \mu, t \in \mathbb{R}$.

Equation (24) can be written as;

$$\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\} = 1-S(t) \tag{26}$$

Substituting equation (26) into equation (25), one gets

$$S'(t) = -\frac{1}{\sigma}\left(\exp\left(-\frac{t-\mu}{\sigma}\right)\right)(1-S(t)) \tag{27}$$

Differentiate equation (27) to have

$$S''(t) = -\frac{1}{\sigma}\left\{-\left(\exp\left(-\frac{t-\mu}{\sigma}\right)\right)S'(t)\right.$$

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$$-\frac{1}{\sigma}\left(\exp\left(-\frac{t-\mu}{\sigma}\right)\right)(1-S(t))\Bigg\} \quad (28)$$

The condition necessary for the existence of equation is $\sigma > 0, \mu, t \in \mathbb{R}$

$$S''(t) = \frac{1}{\sigma}\exp\left(-\frac{t-\mu}{\sigma}\right)\left\{S'(t) + \frac{1}{\sigma}(1-S(t))\right\} \quad (29)$$

Equation (27) can be simplified to become;

$$\frac{1}{\sigma}\left(\exp\left(-\frac{t-\mu}{\sigma}\right)\right) = -\frac{S'(t)}{1-S(t)} \quad (30)$$

Substituting equation (30) into equation (29) yields

$$S''(t) = -\frac{S'(t)}{1-S(t)}\left(S'(t) + \frac{1}{\sigma}(1-S(t))\right) \quad (31)$$

The second order ordinary differential equation for the survival function of the Gumbel distribution is given by;

$$\sigma(1-S(t))S''(t) + \sigma S'^2(t) + (1-S(t))S'(t) = 0 \quad (32)$$

$$S(0) = 1-\exp\left\{-\left[\exp\left(\frac{\mu}{\sigma}\right)\right]\right\} \quad (33)$$

$$S'(0) = -\frac{1}{\sigma}\left(\exp\left(\frac{\mu}{\sigma}\right)\right)\exp\left\{-\left[\exp\left(\frac{\mu}{\sigma}\right)\right]\right\} \quad (34)$$

V. Inverse Survival Function

The inverse survival function of the Gumbel distribution is given as;

$$Q(p) = \mu - \sigma\ln(-\ln(1-p)) \quad (35)$$

To obtain the first order ordinary differential equation for the inverse survival function of the Gumbel distribution, differentiate equation (35), to obtain;

$$Q'(p) = \frac{\sigma}{(1-p)\ln(1-p)} \quad (36)$$

The condition necessary for the existence of equation is $\sigma > 0, 0 < p < 1$.

Differentiating equation (36), we obtain;

$$Q''(p) = \left[\frac{\sigma}{(1-p)^2(\ln(1-p))^2} + \frac{\sigma}{(1-p)^2\ln(1-p)}\right] \quad (37)$$

The condition necessary for the existence of equation is $\sigma > 0, 0 < p < 1$.

Squaring both sides of equation (36) leads to

$$Q'^2(p) = \frac{\sigma^2}{(1-p)^2(\ln(1-p))^2} \quad (38)$$

$$\frac{Q'^2(p)}{\sigma} = \frac{\sigma}{(1-p)^2(\ln(1-p))^2} \quad (39)$$

Also dividing both sides of equation (36) by $1-p$;

$$\frac{Q'(p)}{p} = \frac{\sigma}{(1-p)^2\ln(1-p)} \quad (40)$$

Substituting equations (39) and (40) into equation (37) gives

$$Q''(p) = \left[\frac{Q'^2(p)}{\sigma} + \frac{Q'(p)}{1-p}\right] \quad (41)$$

The second order ordinary differential equation for the inverse survival function of the Gumbel distribution is given by;

$$\sigma(1-p)Q''(p) - (1-p)Q'^2(p) - \sigma Q'(p) = 0 \quad (42)$$

$$Q(0.1) = \mu + 2.25\sigma \quad (43)$$

$$Q'(0.1) = -10.5458\sigma \quad (44)$$

VI. Hazard Function

The hazard function of the Gumbel distribution is given as;

$$h(t) = \frac{\frac{1}{\sigma}\exp\left\{-\left[\frac{t-\mu}{\sigma} + \exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}}{1-\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}} \quad (45)$$

To obtain the first order ordinary differential equation for the hazard function of the Gumbel distribution, differentiate equation (45), to obtain;

$$\frac{1}{\sigma}\left(1-\exp\left(-\frac{t-\mu}{\sigma}\right)\right)$$

$$h'(t) = -\frac{\exp\left\{-\left[\frac{t-\mu}{\sigma} + \exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}}{\exp\left\{-\left[\frac{t-\mu}{\sigma} + \exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}}h(t)$$

$$\frac{1}{\sigma}\exp\left\{-\left[\frac{t-\mu}{\sigma} + \exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\} \quad (46)$$

$$+\frac{\left(1-\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}\right)^{-2} h(t)}{\left(1-\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}\right)^{-1}}$$

$$h'(t) = \left\{-\frac{1}{\sigma}\left(1-\exp\left(-\frac{t-\mu}{\sigma}\right)\right) + \frac{\frac{1}{\sigma}\exp\left\{-\left[\frac{t-\mu}{\sigma} + \exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}}{\left(1-\exp\left\{-\left[\exp\left(-\frac{t-\mu}{\sigma}\right)\right]\right\}\right)}\right\}h(t) \quad (47)$$

The condition necessary for the existence of equation is $\sigma > 0, \mu, t \in \mathbb{R}$.

$$h'(t) = \left\{-\frac{1}{\sigma}\left(1-\exp\left(-\frac{t-\mu}{\sigma}\right)\right) + h(t)\right\}h(t) \quad (48)$$

Differentiating equation (48), one gets

$$h''(t)=\left\{-\frac{1}{\sigma}\left(1-\exp\left(-\frac{t-\mu}{\sigma}\right)\right)+h(t)\right\}h'(t) + \left\{-\frac{1}{\sigma^2}\left(\exp\left(-\frac{t-\mu}{\sigma}\right)\right)+h'(t)\right\}h(t) \quad (50)$$

The condition necessary for the existence of equation is $\sigma>0, \mu, t\in\mathbb{R}$.

The following equations obtained from the simplification of equation (48) are needed to simplify equation (50);

$$\frac{h'(t)}{h(t)}=-\frac{1}{\sigma}\left(1-\exp\left(-\frac{t-\mu}{\sigma}\right)\right)+h(t) \quad (51)$$

$$\frac{h'(t)}{h(t)}-h(t)=-\frac{1}{\sigma}\left(1-\exp\left(-\frac{t-\mu}{\sigma}\right)\right) \quad (52)$$

$$-\sigma\left(\frac{h'(t)}{h(t)}-h(t)\right)=1-\exp\left(-\frac{t-\mu}{\sigma}\right) \quad (53)$$

$$1+\sigma\left(\frac{h'(t)}{h(t)}-h(t)\right)=\exp\left(-\frac{t-\mu}{\sigma}\right) \quad (54)$$

Substituting equations (51) and (54) into equation (50) gives

$$h''(t)=\frac{h'^2(t)}{h(t)}+ \left\{-\frac{1}{\sigma^2}\left(1+\sigma\left(\frac{h'(t)}{h(t)}-h(t)\right)\right)+h'(t)\right\}h(t) \quad (55)$$

$$h''(t)=\frac{h'^2(t)}{h(t)}-\frac{h(t)}{\sigma^2}-\frac{h'(t)}{\sigma}+\frac{h^2(t)}{\sigma}+h(t)h'(t) \quad (56)$$

The second order ordinary differential equation for the hazard function of the Gumbel distribution is given by;

$$\sigma^2 h(t)h''(t)-\sigma^2 h'^2(t)+(\sigma h(t) -\sigma^2 h^2(t))h'(t)+h^2(t)-\sigma h^3(t)=0 \quad (57)$$

$$h(0)=\frac{\frac{1}{\sigma}\exp\left\{-\left[-\frac{\mu}{\sigma}+\exp\left(\frac{\mu}{\sigma}\right)\right]\right\}}{1-\exp\left\{-\left[\exp\left(\frac{\mu}{\sigma}\right)\right]\right\}} \quad (58)$$

$$h'(0)=\left\{-\frac{1}{\sigma}\left(1-\exp\left(\frac{\mu}{\sigma}\right)\right)+h(0)\right\}h(0) \quad (59)$$

VII. Reversed Hazard Function

The reversed hazard function of the Gumbel distribution is given as;

$$j(t)=\frac{1}{\sigma}\exp\left(-\frac{t-\mu}{\sigma}\right) \quad (60)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the Gumbel distribution, differentiate equation (60), to obtain;

$$j'(t)=-\frac{1}{\sigma^2}\exp\left(-\frac{t-\mu}{\sigma}\right) \quad (61)$$

The condition necessary for the existence of equation is $\sigma>0, \mu, t\in\mathbb{R}$.

$$\sigma j'(t)=-\frac{1}{\sigma}\exp\left(-\frac{t-\mu}{\sigma}\right) \quad (62)$$

$$\sigma j'(t)=-j(t) \quad (63)$$

The first order ordinary differential equation for the reversed hazard function of the Gumbel distribution is given by;

$$\sigma j'(t)+j(t)=0 \quad (64)$$

$$j(0)=\frac{1}{\sigma}\exp\left(\frac{\mu}{\sigma}\right) \quad (65)$$

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [34-48]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the Gumbel distribution. The work was simplified by the application of simple algebraic procedures. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

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