Classes of Ordinary Differential Equations Obtained for the Probability Functions of 3-Parameter Weibull Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the 3-parameter Weibull distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are a new ways of understanding the nature of the probability functions that characterize the distribution.**

***Index Terms*— 3-parameter Weibull distribution, differential calculus, probability density function, survival function, quantile function.**

I. Introduction

THE 3-parameter Weibull distribution is a variant of the Weibull distribution and was obtained to improve the flexibility of modeling with Weibull distribution [1]. The distribution has been studied by [2], where they estimated the shape parameter of the distribution. Cran [3] studied extensively the properties of moment estimators of the distribution while [4] proposed the robust estimator for the 3-parameter Weibull distribution. Some other aspects that have been studied includes: conditional expectation [5], parameter estimation under defined censoring [6-7], censoring sampling [8], posterior analysis and reliability [9-10], minimum and robust minimum distance estimation [11-12], three-parameter Weibull equations [13], confidence limits [14], quantile based point estimate of the parameters [15], percentile estimation [16], methods of estimation of parameters [17-21]. Strong computational techniques have now been used in the estimation of parameters of the distribution such as particle swarm optimization [22], differential evolution [23]. Li [24] applied the least square method in the estimation of the parameters of the distribution. Mahmoud [25] observed that the 3-parameter inverse Gaussian distribution can be used and apply as an alternative model for the 3-parameter Weibull distribution. The distribution has been used in the modeling of real life situations such as: fatigue crack growth [26], step-stress accelerated life test [27], ageing [28], helicopter blade reliability [29], cost estimation [30], time between failures of machine tools [31].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of 3-parameter Weibull distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [32], beta distribution [33], raised cosine distribution [34], Lomax distribution [35], beta prime distribution or inverted beta distribution [36].

II. Probability Density Function

The probability density function of the 3- parameter Weibull distribution is given as;

$$f(x)=\frac{\beta}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1}\mathrm{e}^{-\left(\frac{x-\alpha}{\eta}\right)^{\beta}} \tag{1}$$

with the parameters $\alpha \in \mathbb{R}, \beta,\eta,>0, x\geq 0$.

To obtain the first order ordinary differential equation for the probability density function of the 3-parameter Weibull distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\frac{\frac{\beta-1}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-2}}{\left(\frac{x-\alpha}{\eta}\right)^{\beta-1}}-\frac{\frac{\beta}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1}\mathrm{e}^{-\left(\frac{x-\alpha}{\eta}\right)^{\beta}}}{\mathrm{e}^{-\left(\frac{x-\alpha}{\eta}\right)^{\beta}}}\right\}f(x) \tag{2}$$

$$f'(x)=\left\{\frac{\beta-1}{x-\alpha}-\frac{\beta}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1}\right\}f(x) \tag{3}$$

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The condition necessary for the existence of the equation is $x, \alpha, \beta, \eta > 0$

The differential equations can only be obtained for particular values of α, β and η.

When $\beta = 1$, equation (3) becomes;

$$f_a'(x) = \left(-\frac{1}{\eta}\right) f_a(x) \quad (4)$$

$$\eta f_a'(x) + f_a(x) = 0 \quad (5)$$

When $\beta = 2$, equation (3) becomes;

$$f_b'(x) = \left\{\frac{1}{x-\alpha} - \frac{2(x-\alpha)}{\eta^2}\right\} f_b(x) \quad (6)$$

$$\eta^2 (x-\alpha) f_b'(x) - (\eta^2 - 2(x-\alpha)^2) f_b(x) = 0 \quad (7)$$

When $\beta = 3$, equation (3) becomes;

$$f_c'(x) = \left\{\frac{2}{x-\alpha} - \frac{3(x-\alpha)^2}{\eta^3}\right\} f_c(x) \quad (8)$$

$$\eta^3 (x-\alpha) f_c'(x) - (2\eta^3 - 3(x-\alpha)^3) f_c(x) = 0 \quad (9)$$

Equation (3) is differentiated to obtain;

$$f''(x) = \left\{\frac{\beta-1}{x-\alpha} - \frac{\beta}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1}\right\} f'(x) - \left\{\frac{\beta-1}{(x-\alpha)^2} - \frac{\beta(\beta-1)}{\eta^2}\left(\frac{x-\alpha}{\eta}\right)^{\beta-2}\right\} f(x) \quad (10)$$

The following equations obtained from (3) are needed to simplify equation (10);

$$\frac{f'(x)}{f(x)} = \frac{\beta-1}{x-\alpha} - \frac{\beta}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1} \quad (11)$$

$$\frac{\beta}{\eta}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1} = \frac{\beta-1}{x-\alpha} - \frac{f'(x)}{f(x)} \quad (12)$$

$$\frac{\beta(\beta-1)}{\eta^2}\left(\frac{x-\alpha}{\eta}\right)^{\beta-1} = \frac{\beta-1}{\eta}\left[\frac{\beta-1}{x-\alpha} - \frac{f'(x)}{f(x)}\right] \quad (13)$$

$$\frac{\beta(\beta-1)}{\eta^2}\left(\frac{x-\alpha}{\eta}\right)^{\beta-2} = \frac{\beta-1}{x-\alpha}\left[\frac{\beta-1}{x-\alpha} - \frac{f'(x)}{f(x)}\right] \quad (14)$$

Substitute equations (11) and (14) into equation (10);

$$f''(x) = \frac{f'^2(x)}{f(x)} - \left\{\begin{array}{l}\dfrac{\beta-1}{(x-\alpha)^2} \\ -\dfrac{\beta-1}{x-\alpha}\left[\dfrac{\beta-1}{x-\alpha} - \dfrac{f'(x)}{f(x)}\right]\end{array}\right\} f(x) \quad (15)$$

$$f''(x) = \frac{f'^2(x)}{f(x)} - \frac{(\beta-1) f(x)}{(x-\alpha)^2} - \frac{(\beta-1)^2 f(x)}{(x-\alpha)^2} + \frac{(\beta-1) f'(x)}{x-\alpha} \quad (16)$$

$$f''(x) = \frac{f'^2(x)}{f(x)} - \frac{\beta(\beta-1) f(x)}{(x-\alpha)^2} + \frac{(\beta-1) f'(x)}{x-\alpha} \quad (17)$$

The condition necessary for the existence of the equation is $x \geq 0, x-\alpha \neq 0, f(x) > 0, \beta, \eta > 0$

The second order ordinary differential equation for the probability density function of the 3-parameter Weibull distribution is given by;

$$(x-\alpha)^2 f''(x) - (x-\alpha)^2 f'^2(x) + \beta(\beta-1) f^2(x) - (\beta-1)(x-\alpha) f(x) f'(x) = 0 \quad (18)$$

$$f(0) = \frac{\beta}{\eta}\left(-\frac{\alpha}{\eta}\right)^{\beta-1} e^{-\left(-\frac{\alpha}{\eta}\right)^{\beta}} \quad (19)$$

$$f'(0) = -\frac{\beta}{\eta}\left(-\frac{\alpha}{\eta}\right)^{\beta-1} \left\{\frac{\beta-1}{\alpha} + \frac{\beta}{\eta}\left(-\frac{\alpha}{\eta}\right)^{\beta-1}\right\} e^{-\left(-\frac{\alpha}{\eta}\right)^{\beta}} \quad (20)$$

III. Quantile Function

The Quantile function of the 3- parameter Weibull distribution is given as;

$$Q(p) = \alpha - \eta(-\ln(1-p))^{\frac{1}{\beta}} \quad (21)$$

The parameters are: $\beta, \eta > 0, 0 < p < 1$.

To obtain the first order ordinary differential equation for the Quantile function of the 3-parameter Weibull distribution, differentiate equation (21), to obtain;

$$Q'(p) = -\frac{\eta}{\beta(1-p)}(-\ln(1-p))^{\frac{1}{\beta}-1} \quad (22)$$

The condition necessary for the existence of the equation is $\beta, \eta > 0, 0 < p < 1$.

Equation (21) can also be written as;

$$-\eta(-\ln(1-p))^{\frac{1}{\beta}} = Q(p) - \alpha \quad (23)$$

Substitute equation (23) into equation (22);

$$Q'(p) = \frac{Q(p)-\alpha}{\beta(1-p)(-\ln(1-p))} \quad (24)$$

Equation (23) can also be simplified as;

$$-\ln(1-p) = \left(\frac{\alpha - Q(p)}{\eta}\right)^{\beta} \quad (25)$$

Substitute equation (25) into equation (24);

$$Q'(p) = \frac{(Q(p)-\alpha)\eta^{\beta}}{\beta(1-p)(\alpha - Q(p))^{\beta}} \quad (26)$$

$$Q'(p) = -\frac{(\alpha - Q(p))^{1-\beta}\eta^{\beta}}{\beta(1-p)} \quad (27)$$

$$Q(0.1) = \alpha - \eta(-\ln(0.9))^{\frac{1}{\beta}} \quad (28)$$

The differential equations can only be obtained for particular values of α, β and η.

When $\beta = 1$, equation (27) becomes;

$$Q_a'(p) = -\frac{\eta}{(1-p)} \quad (29)$$

$$(1-p)Q_a'(p) + \eta = 0 \quad (30)$$

When $\beta = 2$, equation (27) becomes;

$$Q_b'(p) = -\frac{\eta^2}{2(1-p)(\alpha - Q_b(p))} \quad (31)$$

$$2(1-p)(\alpha - Q_b(p))Q_b'(p) + \eta^2 = 0 \quad (32)$$

When $\beta = 3$, equation (27) becomes;

$$Q_c'(p) = -\frac{\eta^3}{3(1-p)(\alpha - Q_c(p))^2} \quad (33)$$

$$3(1-p)(\alpha - Q_c(p))^2 Q_c'(p) + \eta^3 = 0 \quad (34)$$

IV. SURVIVAL FUNCTION

The survival function of the 3- parameter Weibull distribution is given as;

$$S(t) = e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}} \quad (35)$$

To obtain the first order ordinary differential equation for the survival function of the 3-parameter Weibull distribution, differentiate equation (35), to obtain;

$$S'(t) = -\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}} \quad (36)$$

The condition necessary for the existence of the equation is $t \geq 0, \alpha \in \mathbb{R}, \beta, \eta > 0$.

$$S'(t) = -\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} S(t) \quad (37)$$

The differential equations can only be obtained for particular values of α, β and η.

When $\beta = 1$, equation (37) becomes;

$$S_a'(t) = -\frac{1}{\eta} S_a(t) \quad (38)$$

$$\eta S_a'(t) + S_a(t) = 0 \quad (39)$$

When $\beta = 2$, equation (37) becomes;

$$S_b'(t) = -\frac{2}{\eta}\left(\frac{t-\alpha}{\eta}\right) S_b(t) \quad (40)$$

$$\eta^2 S_b'(t) + 2(t-\alpha)S_b(t) = 0 \quad (41)$$

When $\beta = 3$, equation (37) becomes;

$$S_c'(t) = -\frac{3}{\eta}\left(\frac{t-\alpha}{\eta}\right)^2 S_c(t) \quad (42)$$

$$\eta^3 S_c'(t) + 3(t-\alpha)^2 S_c(t) = 0 \quad (43)$$

Equation (37) is differentiated in order to obtain a simplified ordinary differential equation;

$$S''(t) = -\frac{\beta}{\eta}\left[\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} S'(t) + \frac{\beta-1}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-2} S(t)\right] \quad (44)$$

$$S''(t) = -\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} S'(t) - \frac{\beta(\beta-1)}{\eta^2}\left(\frac{t-\alpha}{\eta}\right)^{\beta-2} S(t) \quad (45)$$

The condition necessary for the existence of the equation is $t, \alpha, \beta, \eta > 0$.

The following equations obtained from (37) are needed to simplify equation (45);

$$-\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} = \frac{S'(t)}{S(t)} \quad (46)$$

$$-\frac{\beta(\beta-1)}{\eta^2}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} = \frac{(\beta-1)}{\eta}\frac{S'(t)}{S(t)} \quad (47)$$

$$-\frac{\beta(\beta-1)}{\eta^2}\left(\frac{t-\alpha}{\eta}\right)^{\beta-2} = \frac{(\beta-1)}{(t-\alpha)}\frac{S'(t)}{S(t)} \quad (48)$$

Substitute equations (46) and (48) into equation (45);

$$S''(t) = \frac{S'^2(t)}{S(t)} - \frac{(\beta-1)S'(t)}{(t-\alpha)S(t)} \quad (49)$$

The second order ordinary differential equation for the survival function of the 3-parameter Weibull distribution is given by;

$$(t-\alpha)S(t)S''(t) - (t-\alpha)S'^2(t) - (\beta-1)S'(t) = 0 \quad (50)$$

$$S(0) = e^{-\left(-\frac{\alpha}{\eta}\right)^{\beta}} \quad (51)$$

$$S'(0) = -\frac{\beta}{\eta}\left(-\frac{\alpha}{\eta}\right)^{\beta-1} e^{-\left(-\frac{\alpha}{\eta}\right)^{\beta}} \quad (52)$$

Alternatively, the ordinary differential equations can be derived using the results obtained from the probability density function.

Equation (3) can be modified using equation (36) to obtain;

$$S''(t) = \left\{\frac{\beta-1}{t-\alpha} - \frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}\right\} S'(t) \quad (53)$$

When $\beta = 1$, equation (53) becomes;

$$S_d''(t) = \left(-\frac{1}{\eta}\right) S_d{}'(t) \quad (54)$$

$$\eta S_d''(t) + S_d'(t) = 0 \quad (55)$$

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When $\beta = 2$, equation (53) becomes;

$$S_e''(t) = \left\{ \frac{1}{t-\alpha} - \frac{2(t-\alpha)}{\eta^2} \right\} S_e'(t) \quad (56)$$

$$\eta^2 (t-\alpha) S_e''(t) - (\eta^2 - 2(t-\alpha)^2) S_e'(t) = 0 \quad (57)$$

When $\beta = 3$, equation (53) becomes;

$$S_f''(t) = \left\{ \frac{2}{t-\alpha} - \frac{3(t-\alpha)^2}{\eta^3} \right\} S_f'(t) \quad (58)$$

$$\eta^3 (t-\alpha) S_f''(t) - (2\eta^3 - 3(t-\alpha)^3) S_f'(t) = 0 \quad (59)$$

V. Inverse Survival Function

The inverse survival function of the 3- parameter Weibull distribution is given as;

$$Q(p) = \alpha + \eta \left(\ln \frac{1}{p} \right)^{\frac{1}{\beta}} \quad (60)$$

To obtain the first order ordinary differential equation for the inverse survival function of the 3-parameter Weibull distribution, differentiate equation (60), to obtain;

$$Q'(p) = -\frac{\eta}{\beta p} \left(\ln \frac{1}{p} \right)^{\frac{1}{\beta}-1} \quad (61)$$

$$Q'(p) = -\frac{\eta \left(\ln \frac{1}{p} \right)^{\frac{1}{\beta}}}{\beta p \left(\ln \frac{1}{p} \right)} \quad (62)$$

The condition necessary for the existence of the equation is $\beta, \eta > 0, 0 < p < 1$.

Equation (60) can also be written as;

$$\eta \left(\ln \frac{1}{p} \right)^{\frac{1}{\beta}} = Q(p) - \alpha \quad (63)$$

$$\ln \frac{1}{p} = \frac{(Q(p)-\alpha)^{\beta}}{\eta^{\beta}} \quad (64)$$

Substitute equations (63) into equation (62);

$$Q'(p) = -\frac{Q(p)-\alpha}{\beta p \left(\ln \frac{1}{p} \right)} \quad (65)$$

Substitute equation (64) into equation (65);

$$Q'(p) = -\frac{\eta^{\beta} (Q(p)-\alpha)}{\beta p (Q(p)-\alpha)^{\beta}} \quad (66)$$

$$Q'(p) = -\frac{\eta^{\beta} (Q(p)-\alpha)^{1-\beta}}{\beta p} \quad (67)$$

$$\beta p Q'(p) + \eta^{\beta} (Q(p)-\alpha)^{1-\beta} = 0 \quad (68)$$

$$Q(0.1) = \alpha + \eta (\ln 10)^{\frac{1}{\beta}} \quad (69)$$

The differential equations can only be obtained for particular values of α, β and η. Some cases are considered and shown in **Table 1.**

Table 1: Some class of ODE for different parameters of the inverse survival function of the 3-parameter Weibull distribution

β	η	α	Ordinary differential equation
1	1	-	$pQ'(p) + 1 = 0$
1	2	-	$pQ'(p) + 2 = 0$
1	3	-	$pQ'(p) + 3 = 0$
2	1	1	$2p(Q(p)-1)Q'(p) - 1 = 0$
2	1	2	$2p(Q(p)-2)Q'(p) - 1 = 0$
2	2	1	$p(Q(p)-1)Q'(p) - 2 = 0$
2	2	2	$p(Q(p)-2)Q'(p) - 2 = 0$

VI. Hazard Function

The hazard function of the 3- parameter Weibull distribution is given as;

$$h(t) = \frac{\beta}{\eta} \left(\frac{t-\alpha}{\eta} \right)^{\beta-1} \quad (70)$$

To obtain the first order ordinary differential equation for the hazard function of the 3-parameter Weibull distribution, differentiate equation (70), to obtain;

$$h'(t) = \frac{\beta(\beta-1)}{\eta^2} \left(\frac{t-\alpha}{\eta} \right)^{\beta-2} \quad (71)$$

The condition necessary for the existence of the equation is $t, \alpha, \beta, \eta > 0$.

$$h'(t) = \frac{(\beta-1)}{\eta} h(t) \quad (72)$$

The first order ordinary differential equation for the hazard function of the 3-parameter Weibull distribution is given by;

$$\eta h'(t) - (\beta-1) h(t) = 0$$

(73) $h(0) = \frac{\beta}{\eta} \left(-\frac{\alpha}{\eta} \right)^{\beta-1}$

(74) To obtain the second order ordinary differential equation for the hazard function of the 3-parameter Weibull distribution, differentiate equation (71);

$$h''(t) = \frac{\beta(\beta-1)(\beta-2)}{\eta^3} \left(\frac{t-\alpha}{\eta} \right)^{\beta-3} \quad (75)$$

The condition necessary for the existence of the equation is $t, \alpha, \beta, \eta > 0$.

Two ordinary differential equations can be obtained from the simplification of equation (75);

ODE 1; Use equation (70) in equation (75);

$$h''(t) = \frac{(\beta-1)(\beta-2)}{(t-\alpha)^2} h(t) \quad (76)$$

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$$(t-\alpha)^2 h''(t)-(\beta-1)(\beta-2)h(t)=0 \quad (77)$$

ODE 2; Use equation (71) in equation (75)

$$h''(t)=\frac{(\beta-2)}{(t-\alpha)}h'(t) \quad (78)$$

$$(t-\alpha)h''(t)-(\beta-2)h'(t)=0 \quad (79)$$

$$h'(0)=\frac{\beta(\beta-1)}{\eta^2}\left(-\frac{\alpha}{\eta}\right)^{\beta-2} \quad (80)$$

To obtain the third order ordinary differential equation for the hazard function of the 3-parameter Weibull distribution, differentiate equation (75);

$$h'''(t)=\frac{\beta(\beta-1)(\beta-2)(\beta-3)}{\eta^4}\left(\frac{t-\alpha}{\eta}\right)^{\beta-4} \quad (81)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\eta>0$.

Three ordinary differential equations can be obtained from the simplification of equation (81);

ODE 1; Use equation (70) in equation (81);

$$h'''(t)=\frac{(\beta-1)(\beta-2)(\beta-3)}{(t-\alpha)^3}h(t) \quad (82)$$

$$(t-\alpha)^3 h'''(t)-(\beta-1)(\beta-2)(\beta-3)h(t)=0 \quad (83)$$

ODE 2; Use equation (71) in equation (81);

$$h'''(t)=\frac{(\beta-2)(\beta-3)}{(t-\alpha)^2}h'(t) \quad (84)$$

$$(t-\alpha)^2 h'''(t)-(\beta-2)(\beta-3)h'(t)=0 \quad (85)$$

ODE 3; Use equation (75) in equation (81);

$$h'''(t)=\frac{(\beta-3)}{(t-\alpha)}h''(t) \quad (86)$$

$$(t-\alpha)h'''(t)-(\beta-3)h''(t)=0 \quad (87)$$

$$h''(0)=\frac{\beta(\beta-1)(\beta-2)}{\eta^3}\left(-\frac{\alpha}{\eta}\right)^{\beta-3} \quad (88)$$

VII. Reversed Hazard Function

The reversed hazard function of the 3- parameter Weibull distribution is given as;

$$j(t)=\frac{\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}}{1-e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}} \quad (89)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the 3-parameter Weibull distribution, differentiate equation (89), to obtain;

$$j'(t)=\left\{\frac{\frac{\beta-1}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-2}}{\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}}-\frac{\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}}{e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}}-\frac{\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}(1-e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}})^{-2}}{(1-e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}})^{-1}}\right\}j(t) \quad (90)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\eta>0$.

$$j'(t)=\left\{\frac{\beta-1}{t-\alpha}-\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}-\frac{\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}}{(1-e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}})}\right\}j(t) \quad (91)$$

$$j'(t)=\left\{\frac{\beta-1}{t-\alpha}-\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}-j(t)\right\}j(t) \quad (92)$$

The differential equations can only be obtained for particular values of α,β and η.

When $\beta=1$, equation (92) becomes;

$$j_a'(t)=\left(-\frac{1}{\eta}-j_a(t)\right)j_a(t) \quad (93)$$

$$\eta j_a'(t)+j_a(t)+\eta j_a^{\,2}(t)=0 \quad (94)$$

When $\beta=2$, equation (92) becomes;

$$j_b'(t)=\left\{\frac{1}{t-\alpha}-\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)-j_b(t)\right\}j_b(t) \quad (95)$$

$$\eta^2(t-\alpha)j_b'(t)+(\beta(t-\alpha)^2-\eta^2)j_b(t)+\eta^2(t-\alpha)j_b^{\,2}(t)=0 \quad (96)$$

Equation (92) is differentiated to obtain;

$$j''(t)=\left\{\frac{\beta-1}{t-\alpha}-\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}-j(t)\right\}j'(t)-\left\{\frac{\beta-1}{(t-\alpha)^2}-\frac{\beta(\beta-1)}{\eta^2}\left(\frac{t-\alpha}{\eta}\right)^{\beta-2}+j'(t)\right\}j(t) \quad (97)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\eta>0$.

The following equations obtained from (92) are needed to simplify equation (97);

$$\frac{j'(t)}{j(t)}=\frac{\beta-1}{t-\alpha}-\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}-j(t) \quad (98)$$

$$\frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}=\frac{\beta-1}{t-\alpha}-\frac{j'(t)}{f(x)}-j(t) \quad (99)$$

$$\frac{\beta(\beta-1)}{\eta^2}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1}=\frac{\beta-1}{\eta}\left[\frac{\beta-1}{t-\alpha}-\frac{j'(t)}{j(t)}-j(t)\right] \quad (100)$$

$$\frac{\beta(\beta-1)}{\eta^2}\left(\frac{t-\alpha}{\eta}\right)^{\beta-2}=\frac{\beta-1}{t-\alpha}\left[\frac{\beta-1}{t-\alpha}-\frac{j'(t)}{j(t)}-j(t)\right] \quad (101)$$

Substitute equations (98) and (101) into equation (97);

$$j''(t)=\frac{j'^2(t)}{j(t)}-\left\{\begin{array}{l}\frac{\beta-1}{(t-\alpha)^2}-\frac{\beta-1}{t-\alpha}\\ \left[\frac{\beta-1}{t-\alpha}-\frac{j'(t)}{j(t)}-j(t)\right]+j'(t)\end{array}\right\}j(t) \quad (102)$$

$$j''(t)=\frac{j'^2(t)}{j(t)}+j(t)j'(t)-\frac{\beta(\beta-1)j(t)}{(t-\alpha)^2}+\frac{(\beta-1)j'(t)}{t-\alpha}+\frac{(\beta-1)j^2(t)}{t-\alpha} \quad (103)$$

The ODEs can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [37-51]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the 3-parameter Weibull distribution. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

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