Proceedings of the World Congress on Engineering and Computer Science 2017 Vol I
WCECS 2017, October 25-27, 2017, San Francisco, USA

Classes of Ordinary Differential Equations Obtained for the Probability Functions of Burr XII and Pareto Distributions

Hilary I. Okagbue, Sheila A. Bishop, *Member, IAENG*, Abiodun A. Opanuga and Muminu O. Adamu

***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of Burr XII and Pareto distributions. This was made easier since later distribution is a special case of the former. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distributions. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distributions.**

***Index Terms*— Burr XII distribution, differential calculus, probability density function, survival function, quantile function, Pareto distribution.**

I. Introduction

THE 2-parameter Burr XII distribution was considered in this research. It is a continuous distribution proposed by Burr [1] but was popularized through the work of [2], where they applied the distribution to model income distributions. Tadikamalla [3] reviewed the distribution and suggested possible relationships with other distributions while a detailed guide on the application was given by [4]. Al-Hussaini [5] proposed the detailed nature of the order statistics. Other aspects of the distribution available includes: Bayesian estimation [6],[7[,[8], parameter estimation in the presence of outliers [9], expected Fisher information [10], Loss function [11], maximum likelihood in the presence of censored samples [12], estimation of parameters using order statistics [13], estimation of parameters under progressive type II censoring [14], application of neural network in the estimation of the parameters [15], minimax estimation of the parameters [16], entropy based parameter estimation [17], explicit closed form for the characteristic function [18], reliability analysis under random censoring [19], estimation with middle censored samples [20], optimal b-robust estimator [21].

Manuscript received June 30, 2017; revised July 31, 2017. This work was sponsored by Covenant University, Ota, Nigeria.

H. I. Okagbue, S. A. Bishop and A.A. Opanuga are with the Department of Mathematics, Covenant University, Ota, Nigeria.
hilary.okagbue@covenantuniversity.edu.ng
sheila.bishop@covenantuniversity.edu.ng
abiodun.opanuga@covenantuniversity.edu.ng
M. O. Adamu is with the Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria

Other variants, sub models, generalizations of the model have been studied by researchers such as: log Burr XII regression model [22], three parameter [23], beta Burr XII distribution [24], Kumaraswamy Burr XII distribution [25].

The Pareto distribution is a special case of Burr XII was also considered. Pareto distribution is hierarchal, skewed, heavy tailed distribution and characterized by scale and shape parameter. The distribution was famously used in the modeling of distribution of wealth. Recent applications include: modeling loss payment data [26], neurophysiology [27], volatility cluster analysis [28], network management [29], transportation [30], wage distribution [31[and modeling flood frequency [32].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Burr XII and Pareto distributions by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [33], beta distribution [34], raised cosine distribution [35], Lomax distribution [36], beta prime distribution or inverted beta distribution [37].

II. Probability Density Function

The probability density function of the Burr XII distribution is given by;

$$f(x)=\frac{ckx^{c-1}}{(1+x^{c})^{k+1}} \qquad (1)$$

When c = 1, the distribution reduces to the Pareto Distribution.

The probability density function can also be written as

$$f(x)=ckx^{c-1}(1+x^{c})^{-(k+1)} \qquad (2)$$

Differentiate equation (2) to obtain;

$$f'(x)=ck\begin{Bmatrix}-(k+1)c(x^{c-1})^{2}(1+x^{c})^{-(k+2)}\\ +(c-1)x^{c-2}(1+x^{c})^{-(k+1)}\end{Bmatrix} \qquad (3)$$

Simplify to obtain;

$$f'(x) = ck\left\{-\frac{(k+1)c(x^{c-1})^2}{(1+x^c)^{(k+2)}} + \frac{(c-1)x^{c-2}}{(1+x^c)^{(k+1)}}\right\} \quad (4)$$

The condition necessary for the existence of the equation is $c, k, x > 0$

When c = 1, equation (4) becomes;

$$f'(x) = -\frac{k(k+1)}{(1+x)^{(k+2)}} \quad (5)$$

The first order ordinary differential equation for the probability density function of the Pareto distribution is given as;

$$(1+x)^{(k+2)} f'(x) + k(k+1) = 0 \quad (6)$$

$$f(1) = \frac{k}{2^{k+1}} \quad (7)$$

When k = 1, 2, n; equation (6) becomes

$$(1+x)^3 f'(x) + 2 = 0 \quad (8)$$

$$(1+x)^4 f'(x) + 6 = 0 \quad (9)$$

$$(1+x)^{(n+2)} f'(x) + n(n+1) = 0 \quad (10)$$

Equation (4) can be simplified further to obtain;

$$f'(x) = f(x)\left\{-\frac{(k+1)cx^{c-1}}{(1+x^c)} + \frac{(c-1)}{x}\right\} \quad (11)$$

Let $$A(x) = \frac{(k+1)cx^{c-1}}{(1+x^c)} + \frac{(c-1)}{x} \quad (12)$$

The first order ordinary differential for the probability density function of the Burr XII distribution is given as;

$$f'(x) - A(x) f(x) = 0 \quad (13)$$

$$f(1) = \frac{ck}{2^{k+1}} \quad (14)$$

III. Quantile Function

The quantile function of the Burr XII distribution is derived from the cumulative distribution function given as:

$$F(x) = 1 - (1+x^c)^{-k} \quad (15)$$

$$Q(p) = \left[\left(\frac{1}{1-p}\right)^{\frac{1}{k}} - 1\right]^{\frac{1}{c}} \quad (16)$$

$$Q(p) = \left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}} \quad (17)$$

Differentiate equation (17), to obtain;

$$Q'(p) = \frac{1}{ck}\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\left(\frac{1}{c}-1\right)} (1-p)^{-\left(\frac{1}{k}+1\right)} \quad (18)$$

$$Q'(p) = \frac{1}{ck}\frac{\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}} - 1\right]}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)} \quad (19)$$

The condition necessary for the existence of the equation is $c, k > 0, 0 \le p < 1$.

When c = 1, equation (19) becomes;

$$Q'(p) = \frac{1}{k}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)} \quad (20)$$

The first order ordinary differential equation for the quantile function of the Pareto distribution is given as;

$$k(1-p)^{\frac{k+1}{k}} Q'(p) - 1 = 0 \quad (21)$$

$$Q(0) = 0 \quad (22)$$

Simplify equation (19) using equation (16), equation (16) becomes;

$$Q^c(p) = \left(\frac{1}{1-p}\right)^{\frac{1}{k}} - 1 \quad (23)$$

$$Q^c(p) + 1 = (1-p)^{-\frac{1}{k}} \quad (24)$$

Substitute equations (17), (23) and (24) into equation (19), to obtain;

$$Q'(p) = \frac{1}{ck}\left(\frac{Q(p)}{Q^c(p)}\right)\left(\frac{Q^c(p)+1}{1-p}\right) \quad (25)$$

$$ck(1-p)Q^c(p)Q'(p) = Q(p)(Q^c(p)+1) \quad (26)$$

The first order ordinary differential for the quantile function of the Burr XII distribution is given as;

$$ck(1-p)Q^c(p)Q'(p) - Q(p)(Q^c(p)+1) = 0 \quad (27)$$

$$Q(0) = 0 \quad (28)$$

To obtain the second order differential equation, differentiate equation (18) to obtain;

$$Q''(p) = \frac{1}{ck}\left\{\begin{array}{l}\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\left(\frac{1}{c}-1\right)}\left(\frac{k+1}{k}\right)(1-p)^{-\left(\frac{1}{k}+2\right)} \\ +(1-p)^{-\left(\frac{1}{k}+1\right)}\left(\frac{1-c}{ck}\right)\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\left(\frac{1}{c}-2\right)} \\ (1-p)^{-\left(\frac{1}{k}+1\right)}\end{array}\right\} \quad (29)$$

$$Q''(p) = \frac{1}{ck}\left\{\begin{array}{l}\left(\frac{k+1}{k}\right)\frac{\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}} - 1\right]}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)^2} \\ +\left(\frac{1-c}{ck}\right)\frac{\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}} - 1\right]^2}\frac{(1-p)^{-\frac{2}{k}}}{(1-p)^2}\end{array}\right\} \quad (30)$$

The condition necessary for the existence of the equation is $c,k>0, 0\le p<1$.

When c = 1, equation (30) becomes;

$$Q''(p)=\frac{1}{k}\left\{\left(\frac{k+1}{k}\right)\frac{(1-p)^{-\frac{1}{k}}}{(1-p)^{2}}\right\} \tag{31}$$

Equation (31) is simplified to obtain the second order differential equation for the quantile function of the Pareto distribution, given as;

$$k^{2}(1-p)^{\frac{2k+1}{k}}Q''(p)-(k+1)=0 \tag{32}$$

$$Q'(0)=\frac{1}{k} \tag{33}$$

Simplifying equation (30) to obtain;

$$Q''(p)=\frac{1}{ck}\frac{\left[(1-p)^{-\frac{1}{k}}-1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}}-1\right]}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)}$$

$$\left\{\left(\frac{k+1}{k}\right)\left(\frac{1}{1-p}\right)+\left(\frac{1-c}{ck}\right)\frac{1}{\left[(1-p)^{-\frac{1}{k}}-1\right]}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)}\right\} \tag{34}$$

The condition necessary for the existence of the equation is $c,k,0\le p<1$

Simplify using equations (16) and (19);

$$Q''(p)=Q'(p)\left\{\left(\frac{k+1}{k}\right)\left(\frac{1}{1-p}\right)+\frac{(1-c)Q'(p)}{Q(p)}\right\} \tag{35}$$

$$\begin{aligned}k(1-p)Q(p)Q''(p)&=(k+1)Q(p)Q'(p)\\&+k(1-c)(1-p)Q'^{2}(p)\end{aligned} \tag{36}$$

The second order ordinary differential for the quantile function of the Burr XII distribution is given as;

$$\begin{aligned}&k(1-p)Q(p)Q''(p)-(k+1)Q(p)Q'(p)\\&-k(1-c)(1-p)Q'^{2}(p)=0\end{aligned} \tag{37}$$

$$Q'(0)=0 \tag{38}$$

IV. SURVIVAL FUNCTION

The survival function of the Burr XII distribution is given as:

$$S(t)=(1+t^{c})^{-k} \tag{39}$$

Differentiate equation (39), to obtain;

$$S'(t)=-ckt^{c-1}(1+t^{c})^{-(k+1)} \tag{40}$$

Equation (40) can also be written as;

$$S'(t)=-ck\frac{t^{c}}{t}\frac{(1+t^{c})^{-k}}{(1+t^{c})} \tag{41}$$

The condition necessary for the existence of the equation is $c,k,t>0$

When c = 1, equation (41) becomes;

$$S'(t)=-k\frac{(1+t)^{-k}}{(1+t)} \tag{42}$$

$$(1+t)^{k+1}S'(t)=-k \tag{43}$$

The first order ordinary differential equation for the survival function of the Pareto distribution is given as;

$$(1+t)^{k+1}S'(t)+k=0 \tag{44}$$

$$S(1)=\frac{1}{2^{k}} \tag{45}$$

When k = 1, 2, n, equation (44) become,

$$(1+t)^{2}S'(t)+1=0 \tag{46}$$

$$(1+t)^{3}S'(t)+2=0 \tag{47}$$

$$(1+t)^{n+1}S'(t)+n=0 \tag{48}$$

Simplify equation (41) using equation (39), to obtain;

$$S'(t)=-ck\frac{t^{c}}{t}\frac{S(t)}{(1+t^{c})} \tag{49}$$

$$S'(t)=-ckB(t)S(t) \tag{50}$$

Where $B(t)=\frac{ckt^{c}}{t(1+t^{c})}$ (51)

The first order ordinary differential equation for the survival function of the Burr XII distribution is given as;

$$S'(t)+ckB(t)S(t)=0 \tag{52}$$

$$S(1)=\frac{1}{2^{k}} \tag{53}$$

To obtain the second order differential equation, differentiate equation (40) to obtain;

$$S''(t)=-ck\left\{\begin{matrix}-(k+1)c(t^{c-1})^{2}(1+t^{c})^{-(k+2)}\\+(c-1)t^{c-2}(1+t^{c})^{-(k+1)}\end{matrix}\right\} \tag{54}$$

$$S''(t)=-\left\{\begin{matrix}-c(k+1)ck\left(\frac{t^{c}}{t}\right)^{2}\frac{(1+t^{c})^{-k}}{(1+t^{c})^{2}}\\+(c-1)ck\frac{t^{c}}{t^{2}}\frac{(1+t^{c})^{-k}}{(1+t^{c})}\end{matrix}\right\} \tag{55}$$

The condition necessary for the existence of the equation is $c,k,t>0$.

When c = 1, equation (55) becomes;

$$S''(t)=(k+1)k\frac{(1+t)^{-k}}{(1+t)^{2}} \tag{56}$$

The second order ordinary differential equation for the survival function of the Pareto distribution is given as;

$$(1+t)^{k+2}S'(t)-k(k+1)=0 \tag{57}$$

$$S'(1)=-\frac{k}{2^{k+1}} \tag{58}$$

When k = 1, 2, n, equation (57) become,

$$(1+t)^{3}S'(t)-2=0 \tag{59}$$

$$(1+t)^{4}S'(t)-6=0 \tag{60}$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$(1+t)^{n+2}S'(t)-n(n+1)=0 \quad (61)$$

Simplify using equation (55), to obtain;

$$S''(t)=-\left\{\frac{c(k+1)t^{c}S'(t)}{t(1+t^{c})}-\frac{(c-1)S'(t)}{t}\right\} \quad (62)$$

$$S''(t)=-\left\{\frac{c(k+1)t^{c}}{t(1+t^{c})}-\frac{(c-1)}{t}\right\}S'(t)=-D(t)S'(t) \quad (63)$$

Where $D(t)=\frac{c(k+1)t^{c}}{t(1+t^{c})}-\frac{(c-1)}{t}$ (64)

The second order ordinary differential equation for the survival function of the Burr XII distribution is given as;

$$S''(t)+D(t)S'(t)=0 \quad (65)$$

$$S(1)=\frac{1}{2^{k}} \quad (66)$$

$$S'(1)=-\frac{ck}{2^{k+1}} \quad (67)$$

V. Inverse Survival Function

The inverse survival function of the Burr XII distribution is given as;

$$Q(p)=\left[p^{-\frac{1}{k}}-1\right]^{\frac{1}{c}} \quad (68)$$

Differentiate equation (68), o obtain;

$$Q'(p)=-\frac{p^{-\left(\frac{1}{k}+1\right)}}{ck}\left[p^{-\frac{1}{k}}-1\right]^{\left(\frac{1}{c}-1\right)} \quad (69)$$

$$Q'(p)=-\frac{p^{-\left(\frac{1}{k}+1\right)}(p^{-\frac{1}{k}}-1)^{\frac{1}{c}}}{ck(p^{-\frac{1}{k}}-1)} \quad (70)$$

The condition necessary for the existence of the equation is $c,k,0\le p<1$.

Substitute equation (68) into equation (70) to obtain;

$$Q'(p)=-\frac{p^{-\left(\frac{1}{k}+1\right)}Q(p)}{ck(p^{-\frac{1}{k}}-1)} \quad (71)$$

$$Q'(p)=-\frac{Q(p)}{ckp^{\left(\frac{1}{k}+1\right)}(p^{-\frac{1}{k}}-1)} \quad (72)$$

$$Q'(p)=-\frac{Q(p)}{ckp(1-p^{\frac{1}{k}})} \quad (73)$$

Equation (68) can be further simplify as;

$$Q^{c}(p)=p^{-\frac{1}{k}}-1 \quad (74)$$

$$p^{-\frac{1}{k}}=Q^{c}(p)+1 \quad (75)$$

$$p^{\frac{1}{k}}=(Q^{c}(p)+1)^{-1} \quad (76)$$

Substitute equation(76) into equation (73);

$$Q'(p)=-\frac{Q(p)}{ckp\left(1-\frac{1}{Q^{c}(p)+1}\right)}=-\frac{Q(p)(Q^{c}(p)+1)}{ckpQ^{c}(p)} \quad (77)$$

$$Q'(p)=-\frac{Q^{1-c}(p)(Q^{c}(p)+1)}{ckp} \quad (78)$$

$$Q'(p)=-\left(\frac{Q(p)+Q^{1-c}(p)}{ckp}\right) \quad (79)$$

The first order ordinary differential equation for the inverse survival function of the Burr XII distribution is given as;

$$ckpQ'(p)+Q(p)+Q^{1-c}(p)=0 \quad (80)$$

$$Q(0)=0 \quad (81)$$

When c = 1, equation (80) becomes;

$$kpQ'(p)+Q(p)+1=0 \quad (82)$$

VI. Hazard Function

The hazard function of the Burr XII distribution is given as:

$$h(t)=\frac{ckt^{c-1}}{1+t^{c}} \quad (83)$$

Differentiate equation (83) to obtain;

$$h'(t)=ck[-c(t^{c-1})^{2}(1+t^{c})^{-2}+(c-1)t^{c-2}(1+t^{c})^{-1}] \quad (84)$$

$$h'(t)=ck\left[-\frac{c(t^{c-1})^{2}}{(1+t^{c})^{2}}+\frac{(c-1)t^{c-2}}{(1+t^{c})}\right] \quad (85)$$

The condition necessary for the existence of the equation is $c,k,t>0$

When c = 1, equation (85) becomes;

$$h'(t)=k\left[-\frac{1}{(1+t)^{2}}\right] \quad (86)$$

The first order ordinary differential equation for the hazard function of the Pareto distribution is given as;

$$(1+t)^{2}h'(t)+k=0 \quad (87)$$

$$h(1)=\frac{k}{2} \quad (88)$$

Simplify equation (85) to obtain;

$$h'(t)=\left[-\frac{ct^{c-1}}{(1+t^{c})}+\frac{(c-1)}{t}\right]h(t) \quad (89)$$

$$h'(t)=\left[-\frac{h(t)}{k}+\frac{(c-1)}{t}\right]h(t) \quad (90)$$

$$kth'(t)=-th^{2}(t)+(c-1)kh(t) \quad (91)$$

The first order ordinary differential equation for the hazard function of the Burr XII distribution is given as;

$$kth'(t)+th^{2}(t)-(c-1)kh(t)=0 \quad (92)$$

$$h(1)=\frac{ck}{2} \quad (93)$$

To obtain the second order differential equation, differentiate equation (84) to obtain;

$$h''(t)=ck\left\{\begin{matrix}2c(t^{c-1})^3(1+t^c)^{-3}-2c(c-1)(t^{c-1})(t^{c-2})\\(1+t^c)^{-2}-c(c-1)(t^{c-1})(t^{c-2})(1+t^c)^{-2}\\+(c-1)(c-2)t^{c-3}(1+t^c)^{-1}\end{matrix}\right\} \quad (94)$$

$$h''(t)=ck\left\{\begin{matrix}\frac{2c(t^{c-1})^3}{(1+t^c)^3}-\frac{2c(c-1)(t^{c-1})(t^{c-2})}{(1+t^c)^2}\\-\frac{c(c-1)(t^{c-1})(t^{c-2})}{(1+t^c)^2}+\frac{(c-1)(c-2)t^{c-3}}{(1+t^c)}\end{matrix}\right\} \quad (95)$$

The condition necessary for the existence of the equation is $c,k,t>0$

When c = 1, equation (95) becomes;

$$h''(t)=k\left\{\frac{2}{(1+t)^3}\right\} \quad (96)$$

The second order ordinary differential equation for the hazard function of the Pareto distribution is given as;

$$(1+t)^3h''(t)-2k=0$$

(97) $$h'(1)=-\frac{k}{2^2}$$

(98) Simplify equation (95) to obtain;

$$h''(t)=\frac{ckt^{c-1}}{1+t^c}\left\{\begin{matrix}\frac{2c(t^{c-1})^2}{(1+t^c)^2}-\frac{2c(c-1)(t^{c-2})}{(1+t^c)}\\-\frac{c(c-1)(t^{c-2})}{(1+t^c)}+\frac{(c-1)(c-2)}{t^2}\end{matrix}\right\} \quad (99)$$

$$h''(t)=h(t)\left\{\begin{matrix}-2c\left[-\frac{c(t^{c-1})^2}{(1+t^c)^2}-\frac{(c-1)(t^{c-2})}{(1+t^c)}\right]\\-\frac{c(c-1)(t^{c-2})}{(1+t^c)}+\frac{(c-1)(c-2)}{t^2}\end{matrix}\right\} \quad (100)$$

$$h''(t)=h(t)\left\{-2c\left(\frac{h'(t)}{ck}\right)-(c-1)\left[\frac{c(t^{c-2})}{(1+t^c)}+\frac{(c-2)}{t^2}\right]\right\} \quad (101)$$

$$h''(t)=-h(t)\left\{\frac{2h'(t)}{k}+(c-1)\left[\frac{h(t)}{kt}+\frac{(c-2)}{t^2}\right]\right\} \quad (102)$$

The second order ordinary differential equation for the hazard function of the Burr XII distribution is given as;

$$kt^2h''(t)+h(t)\left[2t^2h'(t)+(c-1)(th(t)+k(c-2)\right]=0 \quad (103)$$

$$h'(1)=ck\left[\frac{c-1}{2}-\frac{c}{4}\right] \quad (104)$$

VII. Reversed Hazard Function

The reversed hazard function of the Burr XII distribution is given as:

$$j(t)=\frac{ckt^{c-1}}{(1+t^c)[(1+t^c)^k-1]} \quad (105)$$

Differentiate equation (105) to obtain;

$$j'(t)=\left\{\begin{matrix}\frac{(c-1)t^{c-2}}{t^{c-1}}-\frac{ct^{c-1}(1+t^c)^{-2}}{(1+t^c)^{-1}}\\-\frac{ckt^{c-1}(1+t^c)^{k-1}[(1+t^c)^k-1]^{-2}}{[(1+t^c)^k-1]^{-1}}\end{matrix}\right\}j(t) \quad (106)$$

The condition necessary for the existence of the equation is $c,k,t>0.$

$$j'(t)=\left\{\frac{(c-1)}{t}-\frac{ct^{c-1}}{(1+t^c)}-\frac{ckt^{c-1}(1+t^c)^k}{(1+t^c)((1+t^c)^k-1)}\right\}j(t) \quad (107)$$

$$j'(t)=\left\{\frac{(c-1)}{t}-\frac{ct^{c-1}}{(1+t^c)}-(1+t^c)^kj(t)\right\}j(t) \quad (108)$$

The ordinary differential equations can be obtained for particular values of the parameters.

When c = 1, equation (108) becomes;

$$j'(t)=-\left\{\frac{1}{1+t}+(1+t)^kj(t)\right\}j(t) \quad (109)$$

The first order ordinary differential equation for the reverse hazard function of the Pareto distribution is given as;

$$(1+t)j'(t)+(1+t)^{k+1}j^2(t)+j(t)=0$$

(110) $$j(1)=\frac{k}{2(2^k-1)}$$

(111)

The ODEs can be obtained for the particular values of the distribution which will require further classifications and analysis. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [38-49]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this paper, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Burr XII and Pareto distributions. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

ACKNOWLEDGEMENT

The authors are unanimous in appreciation of financial sponsorship from Covenant University. The constructive suggestions of the reviewers are greatly appreciated.

REFERENCES

[1] I.W. Burr, "Cumulative frequency functions", *Ann. Math. Stat.*, vol. 13, no. 2, pp. 215–232, 1942.
[2] S. Singh, S. and G. Maddala, "A Function for the Size Distribution of Incomes",*Econometrica*, vol. 44, no. 5, pp. 963–970, 1976.
[3] P.R. Tadikamalla, "A Look at the Burr and Related Distributions", *Int. Stat. Rev.*, vol. 48, no. 3, pp. 337–344, 1980.
[4] R.N. Rodriguez, "A guide to Burr Type XII distribution", *Biometrika*. vol. 64, no. 1, pp. 129–134, 1977.
[5] E.K. Al-Hussaini, "A characterization of the Burr type XII distribution", *Appl. Math. Lett.*, vol. 4, no. 1, pp. 59-61, 1991.
[6] M.A. Mousa and Z.F. Jaheen, "Bayesian prediction for the Burr type XII model based on doubly censored data", *Statistics*, vol. 29, no. 3, pp. 285-294, 1997.
[7] A.J. Watkins, "An algorithm for maximum likelihood estimation in the three parameter Burr XII distribution", *Comput. Stat. Data Analy.*, vol. 32, no. 1, pp. 19-27, 1999.
[8] M.Y. Danish and M. Aslam, "Bayesian analysis of censored Burr XII distribution", *Elec. J. Appl. Stat. Analy.*, vol. 7, no. 2, pp. 326-342, 2014.
[9] A.M. Hossain and S.K. Nath, "Estimation of parameters in the presence of outliers for a Burr XII distribution", *Comm. Stat. Theo. Meth.*, vol. 26, no. 3, pp. 637-652, 1997.
[10] A.J. Watkins, "A note on expected fisher information for the Burr XII distribution", *Microel. Relia.*, vol. 37, no. 12, pp. 1849-1852, 1997.
[11] D. Moore and A.S. Papadopoulos, "The Burr type XII distribution as a failure model under various loss functions", *Microel. Relia.*, vol. 40, no. 12, pp. 2117-2122, 1999.
[12] F.K. Wang, J.B. Keats and .B. Zimmer, "Maximum likelihood estimation of the burr XII parameters with censored and uncensored data", *Microel. Relia.*, vol. 36, no. 3, pp. 359-362, 1996.
[13] I. Malinowska, P. Pawlas and D. Szynal, "Estimation of location and scale parameters for the Burr XII distribution using generalized order statistics", *Linear Algebra Appli.*, vol. 417, no. 1, pp. 150-162, 2006.
[14] Y.L. Lio, T.R. Tsai and J.Y. Chiang, "Parameter estimations for the burr type XII distribution under progressive type II censoring", *ICIC Express Lett.*, vol. 3, no. 4, pp. 1459-1463, 2009.
[15] B. Abbasi, S.Z. Hosseinifard and D.W. Coit, "A neural network applied to estimate Burr XII distribution parameters", *Relia. Engine. Syst. Safety*, vol. 95, no. 6, pp. 647-654, 2010.
[16] M. Yarmohammadi and H. Pazira, "Minimax estimation of the parameter of the Burr type XII distribution", *J. Appl. Sci. Res.*, vol. 6, no. 12, pp. 6611-6622, 2010.
[17] Z. Hao and V.P. Singh, "Entropy-based parameter estimation for extended Burr XII distribution", *Stoch. Environ. Res. Risk Assess.*, vol. 23, no. 8, pp. 1113-1122, 2009.
[18] S. Nadarajah, T.K. Pogány and R.K. Saxena, "On the characteristic function for Burr distributions. *Statistics*, vol. 46, no. 3, pp. 419-428, 2012.
[19] X.L. Shi, "Reliability analysis for the Burr XII units under random censoring", *Appl. Mech. Mater.*, vol. 321-324, pp. 2265-2268, 2013.
[20] A.H. Abuzaid, "The estimation of the Burr-XII parameters with middle-censored data", *SpringerPlus*, vol. 4, no. 1, Article 4:101, 2015.
[21] F.Z. Doğru, F.Z. and O. Arslan, "Optimal B-robust estimators for the parameters of the Burr XII distribution. *J. Stat. Comput. Simul.*, vol. 86, no. 6, pp. 1133-1149, 2016.
[22] G.O. Silva, E.M.M. Ortega, V.G. Cancho and M.L. Barreto, "Log-Burr XII regression models with censored data", *Comput. Stat. Data Analy.*, vol. 52, no. 7, pp. 3820-3842, 2008.
[23] A.J. Watkins, "On the likelihood function for the three parameter Burr XII distribution", *Int. J. Relia. Qual. Safety Engine.*, vol. 8, no. 2, pp. 173-188, 2001.
[24] P.F. Paranaíba, E.M.M. Ortega, G.M Cordeiro and R.R. Pescim, "The beta Burr XII distribution with application to lifetime data", *Comput. Stat. Data Analy.*, vol. 55, no. 2, pp. 1118-1136, 2011.
[25] P.F. Paranaíba, E.M.M. Ortega, G.M. Cordeiro and M.A.D. Pascoa, "The Kumaraswamy Burr XII distribution: theory and practice", *J. Stat Comput Simul.*, vol. 83, no. 11, pp. 2117-2143, 2013.
[26] N.C. Mdziniso and K. Cooray, "Odd Pareto families of distributions for modeling loss payment data", *Scand. Actua. J.*, in press, 2017.
[27] Z. Liu, J.G. Holden, and R.A. Serota, "Probability density of response times and neurophysiology of cognition", *Adv. Complex Sys.*, vol. 19, no. 4-5, art. 1650013, 2016.
[28] K. Mundnich and M.E. Orchard, "Early online detection of high volatility clusters using Particle Filters", *Expert Sys. Appl.*, vol. 54, pp. 228-240, 2016.
[29] G. Mokryani, A. Majumdar and B.C. Pal, "Probabilistic method for the operation of three-phase unbalanced active distribution networks", *IET Renew. Power Gen.*, vol. 10, no. 7, pp. 944-954, 2016.
[30] X.C. Jiang and Y.S. Zeng, "Research on step punishment intensity model of illegal parking", *China J. High. Trans.*, vol. 29, no. 7, 1 July 2016, Pages 124-133, 2016.
[31] L. Marek and M. Vrabec, "Using mixture density functions for modelling of wage distributions", *Cent. Euro. J. Oper. Res.*, vol. 24, no. 2, pp. 389-405.
[32] D.L. Kriebel, J.D. Geiman and G.R. Henderson, "Future flood frequency under sea-level rise scenarios", *J. Coastal Res.*, vol. 31, no. 5, pp. 1078-1083.
[33] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous univariate distributions, Wiley New York. ISBN: 0-471-58495-9, 1994.
[34] W.P. Elderton, Frequency curves and correlation, Charles and Edwin Layton. London, 1906.
[35] H. Rinne, Location scale distributions, linear estimation and probability plotting using MATLAB, 2010.
[36] N. Balakrishnan and C.D. Lai, Continuous bivariate distributions, 2nd edition, Springer New York, London, 2009.
[37] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 2. 2nd edition, Wiley, 1995.
[38] S.O. Edeki, H.I. Okagbue , A.A. Opanuga and S.A. Adeosun, "A semi - analytical method for solutions of a certain class of second order ordinary differential equations", *Applied Mathematics*, vol. 5, no. 13, pp. 2034 – 2041, 2014.
[39] S.O. Edeki, A.A Opanuga and H.I Okagbue, "On iterative techniques for numerical solutions of linear and nonlinear differential equations", *J. Math. Computational Sci*., vol. 4, no. 4, pp. 716-727, 2014.
[40] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, G.O. Akinlabi, A.S. Osheku and B. Ajayi, "On numerical solutions of systems of ordinary differential equations by numerical-analytical method", *Appl. Math. Sciences*, vol. 8, no. 164, pp. 8199 – 8207, 2014.
[41] S.O. Edeki , A.A. Opanuga, H.I. Okagbue , G.O. Akinlabi, S.A. Adeosun and A.S. Osheku, "A Numerical-computational technique for solving transformed Cauchy-Euler equidimensional equations of homogenous type. *Adv. Studies Theo. Physics*, vol. 9, no. 2, pp. 85 – 92, 2015.
[42] S.O. Edeki , E.A. Owoloko , A.S. Osheku , A.A. Opanuga , H.I. Okagbue and G.O. Akinlabi, "Numerical solutions of nonlinear biochemical model using a hybrid numerical-analytical technique", *Int. J. Math. Analysis*, vol. 9, no. 8, pp. 403-416, 2015.
[43] A.A. Opanuga , S.O. Edeki , H.I. Okagbue and G.O. Akinlabi, "Numerical solution of two-point boundary value problems via differential transform method", *Global J. Pure Appl. Math.*, vol. 11, no. 2, pp. 801-806, 2015.
[44] A.A. Opanuga, S.O. Edeki, H.I. Okagbue and G. O. Akinlabi, "A novel approach for solving quadratic Riccati differential equations", *Int. J. Appl. Engine. Res.*, vol. 10, no. 11, pp. 29121-29126, 2015.
[45] A.A Opanuga, O.O. Agboola and H.I. Okagbue, "Approximate solution of multipoint boundary value problems", *J. Engine. Appl. Sci.*, vol. 10, no. 4, pp. 85-89, 2015.
[46] A.A. Opanuga, O.O. Agboola, H.I. Okagbue and J.G. Oghonyon, "Solution of differential equations by three semi-analytical techniques", *Int. J. Appl. Engine. Res.*, vol. 10, no. 18, pp. 39168-39174, 2015.
[47] A.A. Opanuga, H.I. Okagbue, S.O. Edeki and O.O. Agboola, "Differential transform technique for higher order boundary value problems", *Modern Appl. Sci.*, vol. 9, no. 13, pp. 224-230, 2015.
[48] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, S.A. Adeosun and M.E. Adeosun, "Some Methods of Numerical Solutions of Singular System of Transistor Circuits", J. *Comp. Theo. Nanosci.*, vol. 12, no. 10, pp. 3285-3289, 2015.
[49] A.A. Opanuga, E.A. Owoloko, H. I. Okagbue, O.O. Agboola, "Finite Difference Method and Laplace Transform for Boundary Value Problems," Lecture Notes Engine. Comp. Sci: Proc. World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 65-69.