Classes of Ordinary Differential Equations Obtained for the Probability Functions of Cauchy, Standard Cauchy and Log-Cauchy Distributions

Hilary I. Okagbue, Abiodun A. Opanuga, *Member, IAENG*, Enahoro A. Owoloko and Muminu O. Adamu

***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function and hazard function of Cauchy, standard Cauchy and log-Cauchy distributions. The distributions are related by logarithmic transformation. The ODEs for the inverse survival function and reversed hazard functions of the distributions were not considered because of their nature and complexity. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distributions. Solutions of these ODEs by using numerous available methods are a new ways of understanding the nature of the probability functions that characterize the distribution.**

***Index Terms*— Cauchy distribution, differential calculus, probability density function, survival function, quantile function, log-Cauchy distribution.**

I. Introduction

CAUCHY or Lorentz distribution is a continuous distribution, characterized by undefined mean, variance, skewness and kurtosis and as a result, no moment generating function. The probability density is stable and can be expressed analytically. The distribution according to [1] is one the examples of no unique maximum likelihood estimators. The distribution is robust when used in estimation of parameters in regression models [2]. The distribution function can be used in the construction of skewed distributions obtained from the normal kernel [3] and Laplace kernel [4]. Huang and Chen [5] proposed the skew-Cauchy distribution. Nadarajah and Kotz [6] proposed the skewed truncated Cauchy distribution while Liu et al. [7] proposed a distribution that is intermediate between the normal and Cauchy distributions. The distribution is used to derive a method for the solution of stochastic programming [8].

Recent modifications of the distribution called the square Cauchy mixture distribution was proposed by [9] and modified Cauchy distribution proposed by [10] are recent successes to improve the flexibility of the Cauchy random variables. The distribution has been applied in modeling real life phenomenon such as: modeling seismic amplitude variation [11], modeling electricity prices [12] and job shop scheduling [13].

On the other hand, log-Cauchy is the distribution whose logarithm is a Cauchy random variable.

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF) and hazard function (HF) of the Cauchy (standard Cauchy as a special case) and log Cauchy distributions by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [14], beta distribution [15], raised cosine distribution [16], Lomax distribution [17], beta prime distribution or inverted beta distribution [18].

II. Cauchy and Standard Cauchy Distributions Probability Density Function

The probability density function of the Cauchy distribution is given by;

$$f(x)=\frac{1}{\pi\sigma\left[1+\left(\frac{x-\mu}{\sigma}\right)^2\right]} \tag{1}$$

The probability density function can also be written as

$$f(x)=\frac{\sigma}{\pi\left[(x-\mu)^2+\sigma^2\right]} \tag{2}$$

$$f(x)=\frac{\sigma}{\pi}\left[(x-\mu)^2+\sigma^2\right]^{-1} \tag{3}$$

When $\mu = 0$ and $\sigma = 1$, equations (1), (2) and (3) reduces to standard Cauchy distribution, given as;

Manuscript received July 24, 2017; revised July 28, 2017. This work was sponsored by Covenant University, Ota, Nigeria.

H. I. Okagbue, A. A. Opanuga and E.A. Owoloko are with the Department of Mathematics, Covenant University, Ota, Nigeria.
hilary.okagbue@covenantuniversity.edu.ng
abiodun.opanuga@covenantuniversity.edu.ng
alfred.owoloko@covenantuniversity.edu.ng
M. O. Adamu is with the Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria.

$$f(x) = \frac{1}{\pi\left[x^2+1\right]} \quad (4)$$

$$f(x) = \frac{1}{\pi}\left[x^2+1\right]^{-1} \quad (5)$$

Differentiate equation (3) to obtain;

$$f'(x) = -\frac{2\sigma}{\pi}(x-\mu)\left[(x-\mu)^2+\sigma^2\right]^{-2} \quad (6)$$

The condition necessary for the existence of the equation is $\sigma > 0$.

$$f'(x) = -\frac{2\sigma(x-\mu)}{\pi\left[(x-\mu)^2+\sigma^2\right]^2} \quad (7)$$

When μ = 0 and σ = 1, equation (7) becomes;

$$f'(x) = -\frac{2x}{\pi(x^2+1)^2} \quad (8)$$

Using equation (4) in equation (8), to obtain;

$$f'(x) = -\frac{2xf(x)}{(x^2+1)} \quad (9)$$

The first order ordinary differential for the probability density function of the standard Cauchy distribution is given as;

$$(x^2+1)f'(x) + 2xf(x) = 0 \quad (10)$$

$$f(0) = \frac{1}{\pi} \quad (11)$$

Simplify equation (7) using equation (4), to obtain;

$$f'(x) = -2f(x)(x-\mu)\frac{\pi f(x)}{\sigma} = -\frac{2\pi(x-\mu)f^2(x)}{\sigma} \quad (12)$$

The first order ordinary differential for the probability density function of the Cauchy distribution is given as;

$$\sigma f'(x) + 2\pi(x-\mu)f^2(x) = 0 \quad (13)$$

$$f(0) = \frac{\sigma}{\pi(\mu^2+\sigma^2)} \quad (14)$$

To obtain the second order differential equation, differentiate equation (6) to obtain;

$$f''(x) = -\frac{2\sigma}{\pi}\begin{Bmatrix} -4(x-\mu)^2\left[(x-\mu)^2+\sigma^2\right]^{-3} \\ +\left[(x-\mu)^2+\sigma^2\right]^{-2} \end{Bmatrix} \quad (15)$$

$$f''(x) = -\frac{2\sigma}{\pi}\left\{-\frac{4(x-\mu)^2}{\left[(x-\mu)^2+\sigma^2\right]^3} + \frac{1}{\left[(x-\mu)^2+\sigma^2\right]^2}\right\} \quad (16)$$

The condition necessary for the existence of the equation is $\sigma > 0$.

When μ = 0 and σ = 1, equation (16) becomes;

$$f''(x) = -\frac{2}{\pi}\left\{-\frac{4x^2}{(x^2+1)^3} + \frac{1}{(x^2+1)^2}\right\} \quad (17)$$

Simplify equation (17) using equations (4) and (9);

$$f''(x) = -\frac{4xf'(x)}{(x^2+1)} + \frac{f'(x)}{x} = -\left(\frac{4x\pi f(x)}{1} + \frac{1}{x}\right)f'(x) \quad (18)$$

The second order ordinary differential for the probability density function of the standard Cauchy distribution is given as;

$$xf''(x) - (1-4\pi x^2 f(x))f'(x) = 0 \quad (19)$$

$$f'(1) = -\frac{1}{2\pi} \quad (20)$$

Simplify equation (16) using equations (2) and (7), to obtain;

$$f''(x) = -\left\{-\frac{8\sigma(x-\mu)^2}{\pi\left[(x-\mu)^2+\sigma^2\right]^3} + \frac{2\sigma}{\pi\left[(x-\mu)^2+\sigma^2\right]^2}\right\} \quad (21)$$

$$f''(x) = -\left\{\frac{4(x-\mu)f'(x)}{\left[(x-\mu)^2+\sigma^2\right]} - \frac{f'(x)}{(x-\mu)}\right\} = -\left\{\frac{4\pi f(x)}{\sigma} - \frac{1}{(x-\mu)}\right\}f'(x) \quad (22)$$

The second order ordinary differential for the probability density function of the Cauchy distribution is given as;

$$\sigma(x-\mu)f''(x) + \left(4\pi(x-\mu)f(x) - \sigma\right)f'(x) \quad (23)$$

$$f'(0) = \frac{2\mu\sigma}{\pi\left[\mu^2+\sigma^2\right]^2} \quad (24)$$

QUANTILE FUNCTION

The Quantile function of the Cauchy distribution is given by;

$$Q(p) = \mu + \sigma\tan\left[\pi(p-\frac{1}{2})\right] \quad (25)$$

When μ = 0 and σ = 1, equation (25) reduces to Quantile function of the standard Cauchy Distribution, given as;

$$Q(p) = \tan\left[\pi(p-\frac{1}{2})\right] \quad (26)$$

Differentiate equation (25) to obtain;

$$Q'(p) = \sigma\pi\sec^2\left[\pi(p-\frac{1}{2})\right] \quad (27)$$

The condition necessary for the existence of the equation is $0 < p < 1$.

Using the trigonometric identity;

$$\sec^2 p = 1 + \tan^2 p \quad (28)$$

Applying equation (28) in equation (27);

$$Q'(p) = \sigma\pi\left(1 + \tan^2[\pi(p-\frac{1}{2})]\right) \quad (29)$$

Equation (25) can also be written as;

$$\frac{Q(p)-\mu}{\sigma} = \tan[\pi(p-\frac{1}{2})] \quad (30)$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\left(\frac{Q(p)-\mu}{\sigma}\right)^2=\tan^2[\pi(p-\frac{1}{2})] \quad (31)$$

Substitute equation (31) into equation (29);

$$Q'(p)=\sigma\pi\left[1+\left(\frac{Q(p)-\mu}{\sigma}\right)^2\right] \quad (32)$$

$$\sigma Q'(p)=\pi\left[\sigma^2+(Q(p)-\mu)^2\right] \quad (33)$$

The first order ordinary differential for the Quantile function of the Cauchy distribution is given as;

$$\sigma Q'(p)-\pi\sigma^2-\pi(Q(p)-\mu)^2=0 \quad (34)$$

$$Q(\frac{1}{2})=\mu \quad (35)$$

When μ = 0 and σ = 1, equation (34) reduces to the first order ordinary differential equation for the Quantile function of the standard Cauchy distribution, given as;

$$Q'(p)-\pi Q(p)^2-\pi=0 \quad (36)$$

$$Q(\frac{1}{2})=0 \quad (37)$$

To obtain the second order differential equation, differentiate equation (27) to obtain;

$$Q''(p)=2\sigma\pi^2\sec^2[\pi(p-\frac{1}{2})]\tan[\pi(p-\frac{1}{2})] \quad (38)$$

The condition necessary for the existence of the equation is $0<p<1$.

$$Q''(p)=2\pi Q'(p)\tan[\pi(p-\frac{1}{2})] \quad (39)$$

Simplify using equation (30), then equation (39) becomes;

$$Q''(p)=2\pi Q'(p)\left(\frac{Q(p)-\mu}{\sigma}\right) \quad (40)$$

The second order ordinary differential for the Quantile function of the Cauchy distribution is given as;

$$\sigma Q''(p)-2\pi Q'(p)\left(Q(p)-\mu\right)=0 \quad (41)$$

$$Q'(\frac{1}{2})=\sigma\pi \quad (42)$$

To obtain the third order differential equation, differentiate equation (38) to obtain;

$$Q'''(p)=2\sigma\pi^2\left\{\begin{array}{l}\pi\sec^4[\pi(p-\frac{1}{2})]\\ +2\pi\sec^2[\pi(p-\frac{1}{2})]\tan^2[\pi(p-\frac{1}{2})]\end{array}\right\} \quad (43)$$

The condition necessary for the existence of the equation is $0<p<1$.

$$Q'''(p)=2\sigma\pi^2\sec^2[\pi(p-\frac{1}{2})]\left\{\begin{array}{l}\pi\sec^2[\pi(p-\frac{1}{2})]\\ +2\pi\tan^2[\pi(p-\frac{1}{2})]\end{array}\right\} \quad (44)$$

$$Q'''(p)=2\pi Q'(p)\left\{\begin{array}{l}\pi\sec^2[\pi(p-\frac{1}{2})]\\ +2\pi\left(\sec^2[\pi(p-\frac{1}{2})]-1\right)\end{array}\right\} \quad (45)$$

$$Q'''(p)=2\pi Q'(p)\left\{3\pi\sec^2[\pi(p-\frac{1}{2})]-2\pi\right\}$$
$$=2\pi Q'(p)\left\{\frac{3Q'(p)}{\sigma}-2\pi\right\} \quad (46)$$

The third order ordinary differential for the Quantile function of the Cauchy distribution is given as;

$$\sigma Q'''(p)-2\pi Q'(p)\left(3Q'(p)-2\pi\sigma\right)=0 \quad (47)$$

$$Q''(\frac{1}{2})=0 \quad (48)$$

SURVIVAL FUNCTION

The Survival function of the Cauchy distribution is given by;

$$S(t)=-\frac{1}{\pi}\arctan\left(\frac{t-\mu}{\sigma}\right)-\frac{1}{2} \quad (49)$$

$$S'(t)=-f(t) \quad (50)$$

$$S'(t)=-\frac{\sigma}{\pi\left[(t-\mu)^2+\sigma^2\right]} \quad (51)$$

The first order ordinary differential for the Survival function of the Cauchy distribution is given as;

$$\pi((t-\mu)^2+\sigma^2)S'(t)+\sigma=0 \quad (52)$$

$$S(0)=-\frac{1}{\pi}\arctan\left(\frac{-\mu}{\sigma}\right)-\frac{1}{2} \quad (53)$$

When μ = 0 and σ = 1, equation (52) reduces to the first order ordinary differential equation for the survival function of the standard Cauchy distribution, given as;

$$\pi(t^2+1)S'(t)+1=0 \quad (54)$$

$$S(0)=0 \quad (55)$$

To obtain the second order differential equation, differentiate equation (50) to obtain;

$$S''(t)=-f'(t)=\frac{2\pi(t-\mu)S'^2(t)}{\sigma} \quad (56)$$

The second order ordinary differential for the Survival function of the Cauchy distribution is given as;

$$\sigma S''(t)-2\pi(t-\mu)S'^2(t)=0 \quad (57)$$

$$S'(t)=-\frac{\sigma}{\pi(\mu^2+\sigma^2)} \quad (58)$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

HAZARD FUNCTION

The Hazard function of the Cauchy distribution is given by;

$$h(t) = -\frac{\sigma}{\left((t-\mu)^2+\sigma^2\right)\left(\arctan(\frac{t-\mu}{\sigma})+\frac{1}{2}\right)} \quad (59)$$

Differentiate equation (60) to obtain;

$$h'(t) = \frac{\sigma}{\left((t-\mu)^2+\sigma^2\right)^2\left(\arctan(\frac{t-\mu}{\sigma})+\frac{1}{2}\right)^2} + \frac{2(t-\mu)\sigma}{\left((t-\mu)^2+\sigma^2\right)^2\left(\arctan(\frac{t-\mu}{\sigma})+\frac{1}{2}\right)} \quad (60)$$

$$h'(t) = h^2(t) - \frac{2(t-\mu)h(t)}{\left((t-\mu)^2+\sigma^2\right)} \quad (61)$$

The first order ordinary differential for the Hazard function of the Cauchy distribution is given as;

$$((t-\mu)^2+\sigma^2)h'(t) - ((t-\mu)^2+\sigma^2)h^2(t) + 2(t-\mu)h(t) = 0 \quad (62)$$

$$h(0) = -\frac{\sigma}{\left(\mu^2+\sigma^2\right)\left(\arctan(\frac{-\mu}{\sigma})+\frac{1}{2}\right)} \quad (63)$$

When μ = 0 and σ = 1, equation (62) reduces to the first order ordinary differential equation for the Hazard function of the standard Cauchy distribution, given as;

$$(t^2+1)h'(t) - (t^2+1)h^2(t) + 2th(t) = 0 \quad (64)$$

$$h(0) = -2 \quad (65)$$

To obtain the second order differential equation, differentiate equation (61) to obtain;

$$h''(t) = 2h(t)h'(t) - \frac{2(t-\mu)h'(t)}{\left((t-\mu)^2+\sigma^2\right)} + \frac{4(t-\mu)^2h(t)}{\left((t-\mu)^2+\sigma^2\right)^2} - \frac{2h(t)}{\left((t-\mu)^2+\sigma^2\right)} \quad (66)$$

The following equations (67 - 71) obtained from equation (61) are needed to simplify equation (66). The equations are as given;

$$h'(t) - h^2(t) = -\frac{2(t-\mu)h(t)}{\left((t-\mu)^2+\sigma^2\right)} \quad (67)$$

$$\frac{h'(t)-h^2(t)}{(t-\mu)} = -\frac{2h(t)}{\left((t-\mu)^2+\sigma^2\right)} \quad (68)$$

$$\frac{h'(t)-h^2(t)}{h(t)} = -\frac{2(t-\mu)}{\left((t-\mu)^2+\sigma^2\right)} \quad (69)$$

$$(h'(t)-h^2(t))^2 = \frac{4(t-\mu)^2h^2(t)}{\left((t-\mu)^2+\sigma^2\right)^2} \quad (70)$$

$$\frac{(h'(t)-h^2(t))^2}{h(t)} = \frac{4(t-\mu)^2h(t)}{\left((t-\mu)^2+\sigma^2\right)^2} \quad (71)$$

Substitute equations (67 – 71) into equation (66), to obtain;

$$h''(t) = 2h(t)h'(t) + \left(\frac{h'(t)-h^2(t)}{h(t)}\right)h'(t) + \frac{(h'(t)-h^2(t))^2}{h(t)} + \frac{h'(t)-h^2(t)}{(t-\mu)} \quad (72)$$

The second order ordinary differential for the Hazard function of the Cauchy distribution is given as;

$$h''(t) - 2h(t)h'(t) - \left(\frac{h'(t)-h^2(t)}{h(t)}\right)h'(t) - \frac{(h'(t)-h^2(t))^2}{h(t)} - \frac{h'(t)-h^2(t)}{(t-\mu)} = 0 \quad (73)$$

$$h'(0) = \frac{\sigma}{\left(\mu^2+\sigma^2\right)^2\left(\arctan(\frac{-\mu}{\sigma})+\frac{1}{2}\right)^2} - \frac{2\mu\sigma}{\left(\mu^2+\sigma^2\right)^2\left(\arctan(\frac{-\mu}{\sigma})+\frac{1}{2}\right)} \quad (74)$$

$$h'(0) = \frac{\sigma}{\left(\mu^2+\sigma^2\right)^2}\left(\frac{1}{\left(\arctan(\frac{-\mu}{\sigma})+\frac{1}{2}\right)^2} - \frac{2\mu}{\left(\arctan(\frac{-\mu}{\sigma})+\frac{1}{2}\right)}\right) \quad (75)$$

III. LOG-CAUCHY DISTRIBUTION

PROBABILITY DENSITY FUNCTION

The probability density function of the log-Cauchy distribution is given by;

$$f(x) = \frac{1}{\pi\sigma x\left[1+\left(\frac{\ln x-\mu}{\sigma}\right)^2\right]} \quad (76)$$

The probability density function can also be written as

$$f(x) = \frac{\sigma}{\pi x\left[(\ln x-\mu)^2+\sigma^2\right]} \quad (77)$$

$$f(x) = \frac{\sigma}{\pi}(x^{-1})\left[(\ln x-\mu)^2+\sigma^2\right]^{-1} \quad (78)$$

When μ = 0 and σ = 1, equations (77) and (78) reduces to standard Log-Cauchy Distribution, given as;

$$f(x) = \frac{1}{\pi x\left[(\ln x)^2+1\right]} \quad (79)$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$f(x)=\frac{1}{\pi}(x^{-1})\left[(\ln x)^2+1\right]^{-1} \tag{80}$$

Differentiate equation (78) to obtain;

$$f'(x)=-\frac{\sigma}{\pi}\left\{\begin{array}{l}\dfrac{2(\ln x-\mu)}{x^2((\ln x-\mu)^2+\sigma^2)^2} \\ +\dfrac{1}{x^2((\ln x-\mu)^2+\sigma^2)}\end{array}\right\} \tag{81}$$

$$f'(x)=-\frac{\sigma}{\pi x((\ln x-\mu)^2+\sigma^2)} \left\{\frac{2(\ln x-\mu)}{x((\ln x-\mu)^2+\sigma^2)}+\frac{1}{x}\right\} \tag{82}$$

$$f'(x)=-f(x)\left\{\frac{2(\ln x-\mu)}{x((\ln x-\mu)^2+\sigma^2)}+\frac{1}{x}\right\} \tag{83}$$

Equation (77) can also be written as;

$$\frac{\pi f(x)}{\sigma}=\frac{1}{x\left[(\ln x-\mu)^2+\sigma^2\right]} \tag{84}$$

Substitute equation (84) into equation (83) to obtain;

$$f'(x)=-f(x)\left\{\frac{2\pi(\ln x-\mu)f(x)}{\sigma}+\frac{1}{x}\right\} \tag{85}$$

The first order ordinary differential for the probability density function of the Log-Cauchy distribution is given as;

$$\sigma x f'(x)+2\pi x(\ln x-\mu)f^2(x)+\sigma f(x)=0 \tag{86}$$

$$f(1)=\frac{\sigma}{\pi\left[\mu^2+\sigma^2\right]} \tag{87}$$

When μ = 0 and σ = 1, equation (86) reduces to the first order ordinary differential equation for the probability density function of the standard Log-Cauchy distribution, given as;

$$xf'(x)+2\pi x(\ln x)f^2(x)+f(x)=0 \tag{88}$$

$$f(1)=\frac{1}{\pi} \tag{89}$$

QUANTILE FUNCTION

The Quantile function of the Log-Cauchy distribution is given by;

$$\ln Q(p)=\mu+\sigma\tan\left[\pi(p-\frac{1}{2})\right] \tag{90}$$

$$Q(p)=e^{\mu+\sigma\tan[\pi(p-\frac{1}{2})]} \tag{91}$$

When μ = 0 and σ = 1, equations (90) and (91) reduce to Quantile functions of the standard Log-Cauchy Distribution, given as;

$$\ln Q(p)=\tan\left[\pi(p-\frac{1}{2})\right] \tag{92}$$

$$Q(p)=e^{\tan[\pi(p-\frac{1}{2})]} \tag{93}$$

Differentiate equation (90) to obtain;

$$\frac{Q'(p)}{Q(p)}=\sigma\pi\sec^2\left[\pi(p-\frac{1}{2})\right] \tag{94}$$

$$\frac{Q'(p)}{Q(p)}=\sigma\pi\left(1+\tan^2[\pi(p-\frac{1}{2})]\right) \tag{95}$$

Equation (90) can also be written as;

$$\frac{\ln Q(p)-\mu}{\sigma}=\tan[\pi(p-\frac{1}{2})] \tag{96}$$

$$\left(\frac{\ln Q(p)-\mu}{\sigma}\right)^2=\tan^2[\pi(p-\frac{1}{2})] \tag{97}$$

Substitute equation (97) into equation (95);

$$\frac{Q'(p)}{Q(p)}=\sigma\pi\left[1+\left(\frac{\ln Q(p)-\mu}{\sigma}\right)^2\right] \tag{98}$$

$$\sigma Q'(p)=\pi Q(p)\left[\sigma^2+(\ln Q(p)-\mu)^2\right] \tag{99}$$

The first order ordinary differential for the Quantile function of the log-Cauchy distribution is given as;

$$\sigma Q'(p)-\pi Q(p)\left[\sigma^2+(\ln Q(p)-\mu)^2\right]=0 \tag{100}$$

$$Q(\frac{1}{2})=e^{\mu} \tag{101}$$

When μ = 0 and σ = 1, equation (100) reduces to the first order ordinary differential equation for the Quantile function of the standard log-Cauchy distribution, given as;

$$Q'(p)-\pi Q(p)\left[1+(\ln Q(p))^2\right]=0 \tag{102}$$

$$Q(\frac{1}{2})=1 \tag{103}$$

SURVIVAL FUNCTION

The survival function of the log-Cauchy distribution is given by;

$$S(t)=-\frac{1}{\pi}\arctan\left(\frac{\ln t-\mu}{\sigma}\right)-\frac{1}{2} \tag{104}$$

$$S'(t)=-f(t) \tag{105}$$

$$S'(t)=-\frac{\sigma}{\pi t\left[(\ln t-\mu)^2+\sigma^2\right]} \tag{106}$$

The first order ordinary differential for the survival function of the log-Cauchy distribution is given as;

$$\pi t((\ln t-\mu)^2+\sigma^2)S'(t)+\sigma=0 \tag{107}$$

$$S(1)=-\frac{1}{\pi}\arctan\left(\frac{-\mu}{\sigma}\right)-\frac{1}{2} \tag{108}$$

When μ = 0 and σ = 1, equation (107) reduces to the first order ordinary differential equation for the survival function of the standard log-Cauchy distribution, given as;

$$\pi t((\ln t)^2+1)S'(t)+1=0 \tag{109}$$

$$S(1)=\frac{1}{2} \tag{110}$$

To obtain the second order differential equation, differentiate equation (106) and using equation (85) to obtain

$$S''(t) = -f'(t) = -\left\{-f(t)\left\{\frac{2\pi(\ln t-\mu)f(t)}{\sigma}+\frac{1}{t}\right\}\right\} \tag{111}$$

$$S''(t) = \left\{-S'(t)\left\{\frac{2\pi(\ln t-\mu)(-S'(t))}{\sigma}+\frac{1}{t}\right\}\right\} \tag{112}$$

$$\sigma t S''(t) = 2\pi t(\ln t-\mu)S'^2(t)+\sigma S'(t) \tag{113}$$

The second order ordinary differential for the Survival function of the log-Cauchy distribution is given as;

$$\sigma t S''(t) - 2\pi t(\ln t-\mu)S'^2(t)-\sigma S'(t)=0 \tag{114}$$

$$S(1) = -\frac{1}{\pi}\arctan\left(\frac{-\mu}{\sigma}\right)-\frac{1}{2} \tag{115}$$

HAZARD FUNCTION

The hazard function of the log-Cauchy distribution is given by;

$$h(t) = -\frac{\sigma}{t\left((\ln t-\mu)^2+\sigma^2\right)\left(\arctan(\frac{\ln t-\mu}{\sigma})+\frac{1}{2}\right)} \tag{116}$$

Differentiate equation (116) to obtain;

$$h'(t) = \frac{\sigma^2}{t^2\left((\ln t-\mu)^2+\sigma^2\right)^2\left(\arctan(\frac{\ln t-\mu}{\sigma})+\frac{1}{2}\right)^2} + \frac{2\sigma(\ln t-\mu)}{t^2\left((t-\mu)^2+\sigma^2\right)^2\left(\arctan(\frac{t-\mu}{\sigma})+\frac{1}{2}\right)} + \frac{\sigma}{t^2\left((\ln t-\mu)^2+\sigma^2\right)\left(\arctan(\frac{\ln t-\mu}{\sigma})+\frac{1}{2}\right)} \tag{117}$$

Using equation (116) in equation (117), to obtain;

$$h'(t) = h^2(t) - \frac{2(\ln t-\mu)h(t)}{t((\ln t-\mu)^2+\sigma^2)}-\frac{h(t)}{t} \tag{118}$$

The first order ordinary differential for the hazard function of the log-Cauchy distribution is given as;

$$\begin{aligned} &t((\ln t-\mu)^2+\sigma^2)h'(t)-t((\ln t-\mu)^2+\sigma^2)h^2(t) \\ &+2(\ln t-\mu)h(t)+((\ln t-\mu)^2+\sigma^2)h(t)=0 \end{aligned} \tag{119}$$

$$h(1) = -\frac{\sigma}{\left(\mu^2+\sigma^2\right)\left(\arctan(\frac{-\mu}{\sigma})+\frac{1}{2}\right)} \tag{120}$$

When μ = 0 and σ = 1, equation (119) reduces to the first order ordinary differential equation for the hazard function of the standard log-Cauchy distribution, given as;

$$\begin{aligned} &t((\ln t)^2+1)h'(t)-t((\ln t)^2+1)h^2(t) \\ &+2(\ln t)h(t)+((\ln t)^2+1)h(t)=0 \end{aligned} \tag{121}$$

$$h(1) = -2 \tag{122}$$

The ODEs can be obtained for the particular values of the distribution which will require further classifications and analysis. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [19-33]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

IV. CONCLUDING REMARKS

In this paper, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), and hazard function (HF) of the Cauchy, standard Cauchy and log-Cauchy distributions. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs. Further research is suggested on the ODE for inverse survival function (ISF) and reversed hazard function respectively of the distributions.

ACKNOWLEDGEMENT

The authors are unanimous in appreciation of financial sponsorship from Covenant University. The constructive suggestions of the reviewers are greatly appreciated.

REFERENCES

[1] S. Dharmadhikari and K. Joag-Dev, “Examples of nonunique maximum likelihood estimators”, *Amer. Stat.*, vol. 39, no. 3, pp. 199-200, 1985.
[2] L. Battha and J. Somogyi, “Robust estimation using the Cauchy probability density”, *Acta Geod. Geophys. Hungarica*, 31(1-2), 5-9. 1996.
[3] S. Nadarajah and S. Kotz, “Skewed distributions generated by the normal kernel”, *Stat. Prob. Lett.*, vol. 65, no. 3, pp. 269-277, 2003.
[4] S. Nadarajah and S. Kotz, “Skewed distributions generated by the Laplace Kernel”, *Amer. J. Math. Magt. Sci.*, vol. 24, no. 3-4, pp. 321-349, 2004.
[5] W.J. Huang and Y.H. Chen, “Generalized skew-Cauchy distribution”, *Stat. Prob. Lett.*, vol. 77, no. 11, pp. 1137-1147, 2007.
[6] S. Nadarajah and S. Kotz, “A Skewed truncated Cauchy distribution with applications in economics”, *Appl. Econ. Lett.*, vol. 14, no. 13, pp. 957-961, 2007.
[7] T. Liu, P. Zhang, .W.S. Dai and M. Xie, “An intermediate distribution between Gaussian and Cauchy distributions”, *Physica A*, vol. 391, no. 22, pp. 5411-5421, 2012.
[8] M.K. Pal and S. Bagh, “Stochastic programming with Cauchy distribution”, *Pak. J. Stat. Opera. Res.*, vol. 11, no. 4, pp. 543-552, 2015.
[9] A. Ito, “Recognition of sounds using square Cauchy mixture distribution”, *IEEE* ICSIP 201627 , Art. nọ 7888359, pp. 726-730, 2017.

[10] L. Tian, Y. Wang, R. Wang and X. Bu, “Modified Cauchy distribution model of high-order passive intermodulation”, *Antennas and Propagation (ISAP),IEEE*, pp. 586-587, 2017.

[11] Z. Zong, K. Li, X. Yin, M. Zhu, J. Du, W. Chen and W. Zhang, “Broadband seismic amplitude variation with offset inversion”, *Geophysics*, vol. 82, no. 3, pp. M43-M53, 2017.

[12] P. D., Domanski and M. Gintrowski, “Alternative approaches to the prediction of electricity prices”, *Int. J. Energy Sector Magt.*, vol. 11, no. 1, pp. 3-27, 2017.

[13] K. Akram, K. Kamal and A. Zeb, “Fast simulated annealing hybridized with quenching for solving job shop scheduling problem”, *Appl. Soft Comput.*, vol. 49, pp. 510-523, 2016.

[14] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous univariate distributions, Wiley New York. ISBN: 0-471-58495-9, 1994.

[15] W.P. Elderton, Frequency curves and correlation, Charles and Edwin Layton. London, 1906.

[16] H. Rinne, Location scale distributions, linear estimation and probability plotting using MATLAB, 2010.

[17] N. Balakrishnan and C.D. Lai, Continuous bivariate distributions, 2nd edition, Springer New York, London, 2009.

[18] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 2. 2nd edition, Wiley, 1995.

[19] S.O. Edeki, H.I. Okagbue , A.A. Opanuga and S.A. Adeosun, “A semi - analytical method for solutions of a certain class of second order ordinary differential equations”, *Applied Mathematics*, vol. 5, no. 13, pp. 2034 – 2041, 2014.

[20] S.O. Edeki, A.A Opanuga and H.I Okagbue, “On iterative techniques for numerical solutions of linear and nonlinear differential equations”, *J. Math. Computational Sci*., vol. 4, no. 4, pp. 716-727, 2014.

[21] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, G.O. Akinlabi, A.S. Osheku and B. Ajayi, “On numerical solutions of systems of ordinary differential equations by numerical-analytical method”, *Appl. Math. Sciences*, vol. 8, no. 164, pp. 8199 – 8207, 2014.

[22] S.O. Edeki , A.A. Opanuga, H.I. Okagbue , G.O. Akinlabi, S.A. Adeosun and A.S. Osheku, “A Numerical-computational technique for solving transformed Cauchy-Euler equidimensional equations of homogenous type. *Adv. Studies Theo. Physics*, vol. 9, no. 2, pp. 85 – 92, 2015.

[23] S.O. Edeki , E.A. Owoloko , A.S. Osheku , A.A. Opanuga , H.I. Okagbue and G.O. Akinlabi, “Numerical solutions of nonlinear biochemical model using a hybrid numerical-analytical technique”, *Int. J. Math. Analysis*, vol. 9, no. 8, pp. 403-416, 2015.

[24] A.A. Opanuga , S.O. Edeki , H.I. Okagbue and G.O. Akinlabi, “Numerical solution of two-point boundary value problems via differential transform method”, *Global J. Pure Appl. Math.*, vol. 11, no. 2, pp. 801-806, 2015.

[25] A.A. Opanuga, S.O. Edeki, H.I. Okagbue and G. O. Akinlabi, “A novel approach for solving quadratic Riccati differential equations”, *Int. J. Appl. Engine. Res.*, vol. 10, no. 11, pp. 29121-29126, 2015.

[26] A.A Opanuga, O.O. Agboola and H.I. Okagbue, “Approximate solution of multipoint boundary value problems”, *J. Engine. Appl. Sci.*, vol. 10, no. 4, pp. 85-89, 2015.

[27] A.A. Opanuga, O.O. Agboola, H.I. Okagbue and J.G. Oghonyon, “Solution of differential equations by three semi-analytical techniques”, *Int. J. Appl. Engine. Res.*, vol. 10, no. 18, pp. 39168-39174, 2015.

[28] A.A. Opanuga, H.I. Okagbue, S.O. Edeki and O.O. Agboola, “Differential transform technique for higher order boundary value problems”, *Modern Appl. Sci.*, vol. 9, no. 13, pp. 224-230, 2015.

[29] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, S.A. Adeosun and M.E. Adeosun, “Some Methods of Numerical Solutions of Singular System of Transistor Circuits”, J. *Comp. Theo. Nanosci.*, vol. 12, no. 10, pp. 3285-3289, 2015.

[30] A.A. Opanuga, H.I. Okagbue, O.O. Agboola, "Irreversibility Analysis of a Radiative MHD Poiseuille Flow through Porous Medium with Slip Condition," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 167-171.

[31] A.A. Opanuga, E.A. Owoloko, O.O. Agboola, H.I. Okagbue, "Application of Homotopy Perturbation and Modified Adomian Decomposition Methods for Higher Order Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 130-134.

[32] A.A. Opanuga, E.A. Owoloko, H. I. Okagbue, O.O. Agboola, "Finite Difference Method and Laplace Transform for Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 65-69.

[33] A.A. Opanuga, E.A. Owoloko, H.I. Okagbue, “Comparison Homotopy Perturbation and Adomian Decomposition Techniques for Parabolic Equations,” Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 24-27.