

# Comparison Homotopy Perturbation and Adomian Decomposition Techniques for Parabolic Equations

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**Abstract**—This paper compares homotopy perturbation and Adomian decomposition techniques for the solution of parabolic equations. Some examples are considered to illustrate the techniques. The results reveal that the two techniques gave closed form of solution and as such considered most suitable for solving heat flow problems.

**Index Terms**— Parabolic equation, Homotopy perturbation technique, Adomian decomposition technique

## I. INTRODUCTION

Several problems in sciences and engineering are modeled by linear and nonlinear partial differential equations. Linear partial differential equations arise in areas like diffusion equation and wave equation Wazwaz [1], while nonlinear partial differential equations occur in fluid dynamics, plasma Physics, quantum field theory, nonlinear wave propagation and nonlinear fiber optics, Zwillinger [2].

Various techniques have been proposed to solve these numerous models, most of these proposed methods are computationally involving and do not converge easily to the exact solution. Some of these methods are Galerkin method [3], finite difference method [4], Exp-function method [5], Variational iteration method [6], Multiscale finite element method [7] and differential transform technique [8-9].

In this work, two techniques are proposed for the solution of heat equations viz: homotopy perturbation method and Adomian decomposition method. Homotopy perturbation method has been found to be an excellent tool in solving various initial-boundary value problems. This technique was introduced by He [10] to overcome the limitations posed by the traditional perturbation technique; assumption of small parameter, which imposed great restrictions on the application of perturbation technique due to the fact that overwhelming majority of the linear and nonlinear problems do not have small parameters. Two, identification of the so-called small parameter requires some special techniques, because wrong choice of this parameter may results to serious problems. In addition, approximate solution by perturbation technique is valid for the small values of the parameters. Various applications of homotopy

perturbation technique include; Biazar et al. [11], Opanuga et al. [12], Chun [13].

The Adomian decomposition method was introduced by George Adomian (1923-1996) for solving linear and nonlinear differential equations. Since then, this method is known as the Adomian Decomposition Method (ADM) Adomian [14-15]. ADM is a universal technique because of its ability to solve various kinds of linear and nonlinear ordinary and partial differential equations. Some researchers who have adopted it are; Adesanya et al. [16], Wazwaz [17], Opanuga et al. [18]. The rest of this paper are organized as follows: In section 2 we present a brief analysis of the methods, while section 3 contains the illustrative examples and the conclusion is presented in section 4

## II. ANALYSIS OF THE METHODS

### Analysis of Homotopy Perturbation method

Considering the following function

$$\psi(u) - g(r) = 0, r \in \Omega \quad (1)$$

$$\gamma(u, \frac{\partial u}{\partial t}) = 0, r \in \Gamma \quad (2)$$

where  $\psi$  is a general differential operations,  $g(r)$  is a known analytic function,  $\gamma$  is a boundary operator and  $\Gamma$  is the boundary of the domain  $\Omega$ . The operator  $\psi$  can be split into linear  $L(u)$  and nonlinear part  $N(u)$ , written as

$$L(u) + N(u) - g(r) = 0 \quad (3)$$

Using homotopy technique, we can construct a homotopy

$$v(r, p) : \Omega \times [0, 1] \rightarrow R \text{ which satisfies}$$

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[\Psi(v) - g(r)] = 0, p \in [0, 1], r \in \Omega \quad (4)$$

$$\text{or} \\ H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - g(r)] = 0 \quad (5)$$

where  $p \in [0, 1]$  is referred to as homotopy parameter, and  $u_0$  is an initial approximation for the solution of equation (1), which satisfies the boundary conditions. It is obvious from (4) and (5) that

$$H(v, 0) = L(v) - L(u_0) = 0 \quad (6)$$

$$H(v, 1) = A(v) - g(r) = 0 \quad (7)$$

It can be assumed that the solution of (4) and (5) can be written as a series in  $p$ :

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$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (8)$$

At  $p = 1$ , yields the approximate solution of equation (1) of the form

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (9)$$

#### Analysis of Adomian decomposition method

Considering the differential equation below in an operator form as

$$Lu + Ru = g \quad (10)$$

In this case  $L$  is mostly the lower order derivative assumed to be invertible,  $R$  is other differential operator while  $g$  is

the source term. Applying  $L^{-1}$  to both sides of equation (10) and imposing the given condition, we have

$$u = h - L^{-1}(Ru) \quad (11)$$

where the function  $h$  represents given conditions and the source term. The standard Adomian decomposition method gives the solution of  $u(x, t)$  by and infinite series of components written as

$$u = \sum_{n=0}^{\infty} u_k \quad (12)$$

where the components  $u_0, u_1, u_2, \dots$  are determined recursively. Substituting (12) into (11) yields

$$\sum_{n=0}^{\infty} u_k = h - L^{-1} \left( R \left( \sum_{n=0}^{\infty} u_k \right) \right) \quad (13)$$

For more illustration, we can express equation (13) as

$$u_0 + u_1 + u_2 + u_3 + \dots = h - L^{-1} \left( R(u_0 + u_1 + u_2 + u_3 + \dots) \right) \quad (14)$$

We then determine the solution by identifying the zeroth component as

$$u_0 = h \quad (15)$$

and the remaining components are written as the recursive relation

$$u_{k+1} = -L^{-1} \left( R(u_k) \right), k \geq 0 \quad (16)$$

so that

$$u_1 = -L^{-1} \left( R(u_0) \right), \text{ for } k = 0 \quad (17)$$

$$u_2 = -L^{-1} \left( R(u_1) \right), \text{ for } k = 1 \quad (18)$$

$$u_3 = -L^{-1} \left( R(u_2) \right), \text{ for } k = 2 \quad (19)$$

$$\vdots$$

### III. NUMERICAL RESULTS

**Example 1:** We consider an inhomogeneous one dimensional heat flow equation

$$u_t = u_{xx} - 6x, 0 < x < \pi, t > 0 \quad (20)$$

the boundary and the initial conditions are written as

$$u(0, t) = 0, u(\pi, t) = \pi^3, t \geq 0, \quad (21)$$

$$u(x, 0) = x^3 + \sin x$$

The exact solution is

$$u(x, t) = x^3 + e^{-t} \sin x \quad (22)$$

#### Solution by Homotopy perturbation method

Applying the convex homotopy method, we obtain

$$u_{10} + p^1 u_{11} + p^2 u_{12} + \dots = x^3 + \sin x + p \int_0^t \left( \frac{\partial^2 u_{10}}{\partial x^2} + p^1 \frac{\partial^2 u_{11}}{\partial x^2} + p^2 \frac{\partial^2 u_{12}}{\partial x^2} + \dots - 6x \right) dt \quad (23)$$

Comparing the coefficients of equal powers of  $p$ , we obtain

$$p^{(10)} : u_{10} = x^3 + \sin x \quad (24)$$

$$p^{(11)} : u_{11} = \int_0^t \left( \frac{\partial^2 u_{10}}{\partial x^2} \right) dt - \int_0^t (6x) dt = -t \sin x \quad (25)$$

$$p^{(12)} : u_{12} = \int_0^t \left( \frac{\partial^2 u_{11}}{\partial x^2} \right) dt = \frac{t^2}{2!} \sin x \quad (26)$$

$$p^{(13)} : u_{13} = \int_0^t \left( \frac{\partial^2 u_{12}}{\partial x^2} \right) dt = -\frac{t^3}{3!} \sin x \quad (27)$$

Collecting the components together, we have

$$u(x, t) = x^3 + \sin x - t \sin x + \frac{t^2}{2!} \sin x - \frac{t^3}{3!} \sin x + \dots \quad (28)$$

$$u(x, t) = x^3 + \sin x \left[ 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right] \quad (29)$$

and the closed form solution can be written as

$$u(x, t) = x^3 + e^{-t} \sin x \quad (30)$$

#### Solution by Adomian decomposition method

Expressing equation (20) in operator form, we have

$$L_t u(x, t) = L_x u(x, t) - 6x \quad (31)$$

The above operators are defined as

$$L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial^2}{\partial x^2} \quad (32)$$

The integral operator of  $L_t^{-1}$  ( $L_t^{-1}$ ) exists and it is written as

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) dt \quad (33)$$

Applying  $L_t^{-1}$  to both sides of equation (32), we obtain

$$u(x, t) = -6xt + L_t^{-1} \left( L_x u(x, t) \right) \quad (34)$$

Imposing the initial condition, yields

$$u(x, t) = -6xt + x^3 + \sin x + L_t^{-1} \left( L_x u(x, t) \right) \quad (35)$$

The decomposition series  $u(x, t)$  is written as

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t) \quad (36)$$

Substituting (36) in (35), we obtain

$$\sum_{k=0}^{\infty} u_k(x,t) = -6xt + x^3 + \sin x + L_t^{-1} \left( L_x \left( \sum_{k=0}^{\infty} u_k(x,t) \right) \right) \quad (37)$$

The zeroth component is

$$u_0(x,t) = -6xt + x^3 + \sin x \quad (38)$$

The recursive relation below gives other components

$$u_{k+1}(x,t) = L_t^{-1} \left( L_x \left( \sum_{k=0}^{\infty} u_k(x,t) \right) \right) \quad (39)$$

$$u_1(x,t) = L_t^{-1} (L_x u_0(x,t)) = t \sin x + 6xt \quad (40)$$

$$u_2(x,t) = L_t^{-1} (L_x u_1(x,t)) = \frac{t^2}{2!} \sin x \quad (41)$$

$$u_3(x,t) = L_t^{-1} (L_x u_2(x,t)) = -\frac{t^3}{3!} \sin x \quad (42)$$

$$u_4(x,t) = L_t^{-1} (L_x u_3(x,t)) = \frac{t^4}{4!} \sin x \quad (43)$$

Then

$$\sum_{k=0}^{\infty} u_k(x,t) = x^3 - 6xt + \sin x - t \sin x + 6xt + \frac{t^2}{2!} \sin x - \frac{t^3}{3!} \sin x + \dots \quad (44)$$

$$= x^3 + \sin x \left[ 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right] \quad (45)$$

$$= x^3 + e^{-t} \sin x \quad (46)$$

**Example 2:** We will also consider a three dimensional heat equation with lateral heat loss

$$u_t = u_{xx} + u_{yy} + u_{zz} - u, 0 < x, y, z < \pi, t > 0 \quad (47)$$

The boundary and initial conditions are stated as

$$\begin{aligned} u(0, y, z, t) &= u(\pi, y, z, t) = 0 \\ u(x, 0, z, t) &= u(x, \pi, z, t) = 0 \\ u(x, y, 0, t) &= u(x, y, \pi, t) = 0 \\ u(x, y, z, t) &= \sin x \sin y \sin z \end{aligned} \quad (48)$$

and theoretical solution is

$$u(x, y, z, t) = e^{-4t} \sin x \sin y \sin z \quad (49)$$

#### Solution by homotopy perturbation method

The complex homotopy method to equation (47) gives

$$\begin{aligned} u_{10} + p^1 u_{11} + p^2 u_{12} + \dots &= \sin x \sin y \sin z + \\ p \int_0^t \left( \frac{\partial^2 u_{10}}{\partial x^2} + p^1 \frac{\partial^2 u_{11}}{\partial x^2} + p^2 \frac{\partial^2 u_{12}}{\partial x^2} + \dots \right) dt &+ \\ \int_0^t \left( \frac{\partial^2 u_{10}}{\partial y^2} + p^1 \frac{\partial^2 u_{11}}{\partial y^2} + p^2 \frac{\partial^2 u_{12}}{\partial y^2} + \dots \right) dt &+ \\ \int_0^t \left( \frac{\partial^2 u_{10}}{\partial z^2} + p^1 \frac{\partial^2 u_{11}}{\partial z^2} + p^2 \frac{\partial^2 u_{12}}{\partial z^2} + \dots \right) dt &+ \\ \int_0^t (u_{10} + p^1 u_{11} + p^2 u_{12} + p^3 u_{13}) dt &\Big| \end{aligned} \quad (50)$$

Equating equal powers of p, we have

$$p^{(10)} : u_{10}(x, y, z, t) = \sin x \sin y \sin z \quad (51)$$

$$p^{(11)} : u_{11} = -4t \sin x \sin y \sin z = -4t \sin x \sin y \sin z \quad (52)$$

$$p^{(12)} : u_{12} = \frac{16t^2}{2!} \sin x \sin y = \frac{(4t)^2}{2!} \sin x \sin y \quad (53)$$

The solution is

$$u(x, y, z, t) = e^{-4t} \sin x \sin y \sin z \quad (54)$$

#### Solution by Adomian decomposition method

In operator form, equation (47) becomes

$$\begin{aligned} L_t u(x, y, z, t) &= L_x u(x, y, z, t) + \\ L_y u(x, y, z, t) &+ L_z u(x, y, z, t) - u \end{aligned} \quad (55)$$

The above operators are defined as

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial^2}{\partial x^2}, \quad L_y = \frac{\partial^2}{\partial y^2}, \quad L_z = \frac{\partial^2}{\partial z^2} \quad (56)$$

The integral operator of  $L_t$  (i.e.  $L_t^{-1}$ ) exists and it is written as

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) dt \quad (57)$$

Applying  $L_t^{-1}$  to both sides of equation (55) with the initial condition, we obtain

$$\begin{aligned} u(x, y, z, t) &= \sin x \sin y \sin z + \\ L_t^{-1} \left( L_x u(x, y, z, t) + L_y u(x, y, z, t) + \right. & \\ \left. L_z u(x, y, z, t) - u(x, y, z, t) \right) &\quad (58) \end{aligned}$$

The decomposition series  $u(x, y, z, t)$  is written as

$$u(x, y, z, t) = \sum_{k=0}^{\infty} u_k u(x, y, z, t) \quad (59)$$

substituting (59) in (58), we obtain

$$\sum_{k=0}^{\infty} u_k(x, y, z, t) = \sin x \sin y \sin z + L_t^{-1} \left( \begin{array}{l} L_x \sum_{k=0}^{\infty} u_k(x, y, z, t) + L_y \sum_{k=0}^{\infty} u_k(x, y, z, t) + \\ L_z \sum_{k=0}^{\infty} u_k(x, y, z, t) - \sum_{k=0}^{\infty} u_k(x, y, z, t) \end{array} \right) \quad (60)$$

The zeroth component is

$$u_0(x, y, z, t) = \sin x \sin y \sin z \quad (61)$$

The recursive relation below gives other components

$$u_{k+1}(x, y, z, t) = L_t^{-1} \left( \begin{array}{l} L_x \left( \sum_{k=0}^{\infty} u_k(x, y, z, t) \right) + \\ L_y \left( \sum_{k=0}^{\infty} u_k(x, y, z, t) \right) + \\ L_z \left( \sum_{k=0}^{\infty} u_k(x, y, z, t) \right) - \sum_{k=0}^{\infty} u_k(x, y, z, t) \end{array} \right) \quad (62)$$

$$u_1(x, y, z, t) = L_t^{-1} (L_x u_0 + L_y u_0 + L_z u_0 - u_0) = -4t \sin x \sin y \sin z \quad (63)$$

$$u_2(x, y, z, t) = L_t^{-1} (L_x u_1 + L_y u_1 + L_z u_1 - u_1) = \frac{16t^2}{2!} \sin x \sin y \sin z \quad (64)$$

The solution is expressed as

$$u(x, y, z, t) = \sin x \sin y \sin z - 4t \sin x \sin y \sin z + \frac{16t^2}{2!} \sin x \sin y \sin z + \dots \quad (65)$$

$$u(x, t, y, z) = \sin x \sin y \sin z \left[ 1 - 4t + \frac{(4t)^2}{2!} + \dots \right] = e^{-4t} \sin x \sin y \sin z \quad (66)$$

#### IV. CONCLUSION

In this work, some examples of heat flow equations are solved by two semi-analytical techniques: homotopy perturbation method and Adomian decomposition method. The two methods are found to be related in application, the computational size is reduced and they gave close form solution. It is worth mentioning that the examples in this work are solved by imposing only the initial conditions, however the solution could still be obtained by applying the boundary conditions. This is the advantage these methods have over other traditional methods like Crank-Nicolson finite difference method and the rest.

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